Proceedings of ICAD2004
The Third International Conference on Axiomatic Design
Seoul – June 21-24, 2004
ICAD-2004-09

UNIFYING AXIOMATIC DESIGN AND ROBUST DESIGN THROUGH THE TRANSFER FUNCTION

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ABSTRACT

Traditionally, large complex system is evolved from an assemblage of components to form a subsystem and from subsystems to system. The approach invariably fails to capture the connectivity between various output measures and input variables. This paper proposes to "connect the dots" by relating output measures to input variables through the transfer function. With the aid of the transfer function, it becomes possible to give a mathematical treatment of Axiomatic Design and robust design. The treatment clarifies such concepts as independence and information content in Axiomatic Design. It extends robust design to design for multiple functional requirements. It unifies Axiomatic Design and robust design. Examples are provided to illustrate the unification.

1 INTRODUCTION

Over the past two decades, the quality of North American cars has improved significantly. Robust design has played a major role in the improvement. However, in spite of the wide success enjoyed by robust design, the full potential of robust design has yet to be realized. This is because its applications thus far have been limited to component design involving single functional requirement (FR). For large complex system such as automobile that involves numerous components and subsystems interacting in complex fashions, it is impractical to *evolve* a large complex system from an assemblage of components, i.e., *bottom up*. Robust components in and of themselves can not guarantee robust system. A large complex system has to be *designed*, not *evolved*, as a robust system *top down*.

Axiomatic Design provides the theory to design a complex system top down, see Reference 1. It deals with multiple FRs, single FR being a special case, and provides the connectivity between various FRs and design parameters (DPs) through the design matrix. It is therefore natural to develop robust system from Axiomatic Design framework. We shall do this in this paper through the introduction of the transfer function. With the aid of transfer function, it becomes possible to give a mathematical treatment of Axiomatic Design and robust design. The treatment helps clarify such concepts as independence and information content in Axiomatic Design. It extends robust design to the design for multiple functional requirements. It unifies Axiomatic Design and robust design.

2 TRANSFER FUNCTION FOR A SINGLE FR

A transfer function relates the output, the functional requirement FR of a design to its input, the design parameters $\{DP_1, DP_2, \ldots DP_m\}$ which are collectively denoted by the vector $\mathbf{DP}^+$,

$$\begin{aligned} FR &= f\,(DP_1, DP_2 \ldots DP_m); \\ &= f\,(\mathbf{DP}) \end{aligned}$$

The notation $f(\bullet)$ denotes the transfer function itself.

For example, a design to achieve a joint of a cast iron rod with aluminum tubing by press fitting will have the transfer function:

$$\begin{aligned} p &= f(R_{Fe}, R_{Al}, C_{Al}, E_{Fe}, \mu_{Fe}, E_{Al}, \mu_{Al}) \\ &= \left[\frac{2E_{Al}(R_{Fe} - R_{Al})}{(R_{Fe} + R_{Al})}\right]\left\{\left[\frac{4C_{Al}^2 + (R_{Fe} + R_{Al})^2}{4C_{Al}^2 - (R_{Fe} + R_{Al})^2}\right] + \frac{E_{Al}}{E_{Fe}}(1 - \mu_{Fe}) + \mu_{Al}\right\}^{-1} \end{aligned}$$

In the above, the FR is the radial pressure p developed at the interface that holds the rod and the tubing together. The design parameters are the inner radius R_{Al} and outer radius C_{Al} of the aluminum tubing, the radius R_{Fe} of the cast iron rod and the Young modulus E and Poisson ratio μ, subscripts *Al* and *Fe* refer to aluminum and cast iron respectively. For μ_{Al}=0.34, E_{Al}= 71, 000Mpa, μ_{Fe}=0.29 and E_{Fe}= 120,000Mpa; the transfer function reduces to

$$\begin{aligned} p &= f(R_{Fe}, R_{Al}, C_{Al}) \\ &= 142{,}000\left[\frac{(R_{Fe} - R_{Al})}{(R_{Fe} + R_{Al})}\right]\left\{\left[\frac{4C_{Al}^2 + (R_{Fe} + R_{Al})^2}{4C_{Al}^2 - (R_{Fe} + R_{Al})^2}\right] + 0.760\right\}^{-1}. \end{aligned}$$

To design for a radial pressure of p = 30Mpa, we may specify

R_{Fe} = 20.938mm,
R_{Al} = 20.890mm,
C_{Al} = 26.000mm.

The above specifications yield the target radial pressure = 30Mpa. However, due to manufacturing error in components' radius of say ±0.025mm, the actual radial pressure developed would be 30±22.120Mpa.

By contrast, the same design with specifications of

R_{Fe} = 22.908mm,
R_{Al} = 20.825mm,
C_{Al} = 26.000mm;

and in the presence of the same manufacturing errors in components radius of ±0.025mm would yield a radial pressure = 30±12.782Mpa. There is a 50% reduction in error of the radial pressure developed. In other words, the second set of specifications yield a design that is on target and at the same time less sensitive to the manufacturing errors. This example illustrates the two issues involved in design; tuning the design for a target value and rendering the design insensitive to errors. A design must be specified in such a way that it is tuned to target and is insensitive, i.e., robust, to errors as well.

The mathematical formulation for robust design is as follow. To tune the design to the target value FR*, we choose a vector **DP*=** $\left\{DP_1^*, DP_2^*, \cdots DP_m^*\right\}$ such that

$$f(\mathbf{DP}^*) = FR^*. \qquad (1)$$

Once the design is tuned to the target value, any random variation in **DP** around **DP***, which we denote as δ**DP,** produces a random FR given by:

$$FR = f\left(\mathbf{DP}^*\right) + \sum_j \left.\frac{\partial f}{\partial DP_j}\right|_{\mathbf{DP}^*} \delta DP_j .$$

In the above, $\left.\frac{\partial f}{\partial DP_j}\right|_{\mathbf{DP}^*} = f_j'\left(\mathbf{DP}^*\right)$ is the sensitivity of the design to errors evaluated at **DP***. The squared errors of the FR is then given by,

$$\left[FR - f\left(\mathbf{DP}^*\right)\right]^2 = \sum_i \sum_j f_i'\left(\mathbf{DP}^*\right) f_j'\left(\mathbf{DP}^*\right) \delta DP_i \delta DP_j$$

(2)

If the transfer function f(**DP***) is a nonlinear function of **DP***, then the sensitivity $f_i'\left(\mathbf{DP}^*\right)$ would be a function of **DP*** as well. One can therefore specify the value of **DP*** such that the design is tuned to target, i.e., f (**DP***) = FR*; and at the same time insensitive to error in **DP,** i.e., $\mathbf{f_j'}\left(\mathbf{DP}^*\right)$ is low. Thus, robust design stated in mathematical terms is the search for **DP*** that minimizes the squared error in Equation (2) subject to the equality constraint of Equation (1). This strategy of exploiting the nonlinearity of the transfer function to achieve a design that is on target and at the same time insensitive to error was first introduced by Dr. G. Taguchi to North America in the mid-1980, see Reference 2. The transfer function and the robust design have since formed the core strategy for implementing Design for Six Sigma now widely practiced in the industries.

3 TRANSFER FUNCTIONS FOR MULTIPLE FR

A design most likely will have multiple FRs involving multiple DPs. There could be interdependence among the FRs arising from their dependence on the same set of DPs. Since Axiomatic Design provides the connectivity between various FRs and DPs through the design matrix, it is most natural to develop transfer functions for multiple FRs from Axiomatic Design viewpoint.

For multiple FRs wherein an FR_i is a function of some, but not all of the DPs, the multiple functional relationships may be expressed as:

$$\begin{aligned} FR_1 &= f_1\left(DP_1, DP_2, DP_4, \cdots DP_n\right) \\ FR_2 &= f_2\left(DP_2, DP_3, DP_5, \cdots DP_{n-1}\right) \\ \vdots \quad & \quad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ FR_n &= f_n\left(DP_1, DP_4, DP_5, \cdots DP_n\right) \end{aligned} \qquad (3)$$

Or

FR = **f** (**DP**)

A convenient representation of above functional relationships, Equation (3), is the matrix notation:

$$\begin{bmatrix} FR_1 \\ FR_2 \\ FR_3 \\ FR_4 \\ FR_5 \\ \vdots \\ FR_{n-1} \\ FR_n \end{bmatrix} = \begin{bmatrix} X & X & O & X & O & \cdots & O & X \\ O & X & X & O & X & \cdots & X & O \\ \vdots & & & & & \cdots & & \vdots \\ \vdots & & & & & \cdots & & \vdots \\ \vdots & & & & & \cdots & & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \vdots & & & & & \cdots & & \vdots \\ X & O & O & X & X & \cdots & O & X \end{bmatrix} \begin{bmatrix} DP_1 \\ DP_2 \\ DP_3 \\ DP_4 \\ DP_5 \\ \vdots \\ DP_{n-1} \\ DP_n \end{bmatrix}$$

Or,

$$\mathbf{FR} = [DM]\ \mathbf{DP} \qquad (4)$$

The "design matrix" $[DM]^{++}$ above serves only to indicate the connectivity, whether an FR is a function (as indicated by

"X") or not a function (as indicated by "O") of a DP. It is not to be construed as a matrix. Nor should Equation (4) be construed as a matrix equation following the rules of algebraic matrix manipulations. However, the structure of the "design matrix" [DM] does indicate the degree of coupling among the FRs through the DPs. It provides a measure of the difficulty with which we can tune a vector **DP** to achieve a target vector **FR***. For example, a diagonal design matrix, i.e., no coupling, implies that each FR_i can be targeted solely by tuning the corresponding DP_i independent of other $DP_j,\ j \neq i$

There are situations wherein **FR** can be expressed as a linear combination of **DP**.

Namely,

$$FR_i = \sum_{k=1}^{n} a_{ik} DP_k \,.$$

Or,

$$\mathbf{FR} = [A]\,\mathbf{DP} \qquad (5)$$

In these situations, the matrix [A] is in fact a matrix; and Equation (5) constitutes the transfer functions for multiple FR. It is a matrix equation following all the rules of algebraic matrix manipulations.

If $\mathbf{DP}^*$ is the value of **DP** that brings **FR** to its target value $\mathbf{FR}^* = \mathbf{f}\,(\mathbf{DP}^*)$, any deviation in **DP** from $\mathbf{DP}^*$, which we denote as $\delta\mathbf{DP}$, yields an **FR** that may be approximated by a Taylor series expansion of the **FR** around **FR***:

$$\mathbf{FR} = \mathbf{f}(\mathbf{DP}^*) + \sum_j \left.\frac{\partial \mathbf{f}}{\partial DP_j}\right|_{\mathbf{DP}^*} \delta DP_j \qquad (6)$$

++ Bracketed upper case quantities are matrices.

The deviation in **FR** from its target value **FR*** which we denote as $\delta\mathbf{FR}$, is then given by:

$$\begin{aligned}\delta\mathbf{FR} &= \mathbf{FR} - \mathbf{f}\left(\mathbf{DP}^*\right)\\ &= \sum_j \left.\frac{\partial \mathbf{f}}{\partial DP_j}\right|_{\mathbf{DP}^*} \delta DP_j \qquad (7)\\ &= [B]\Delta\mathbf{DP}.\end{aligned}$$

And, the squared error of **FR** is:

$$\delta\mathbf{FR}^T \delta\mathbf{FR} = \delta\mathbf{DP}^T [B]^T [B] \delta\mathbf{DP} \qquad (8)$$

The [B] matrix evaluated at **DP*** is as follows.

$$[B] = \left.\frac{\partial \mathbf{f}}{\partial DP_j}\right|_{\mathbf{DP}^*} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial DP_1} & \frac{\partial f_1}{\partial DP_2} & \cdots\ \cdots & \frac{\partial f_1}{\partial DP_n} \\ \frac{\partial f_2}{\partial DP_1} & \frac{\partial f_2}{\partial DP_2} & \cdots\ \cdots & \frac{\partial f_2}{\partial DP_n} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial DP_1} & \frac{\partial f_n}{\partial DP_2} & \cdots\ \cdots & \frac{\partial f_n}{\partial DP_n} \end{bmatrix}\right|_{\mathbf{DP}^*}$$

(9)

Based on Equations (5), (6) and (7), we may express **FR** as:

$$\mathbf{FR} = [A]\mathbf{DP}^* + [B]\delta\mathbf{DP} \qquad (10)$$

Equation (10) above, derived from the Taylor series expansion of the transfer functions, expresses succinctly the two issues in the design for multiple FR. The first term on the RHS of the equation relates to the tuning of the design for multiple target values of the **FR**. The ease of tuning is dictated by the structure of the [A] matrix, a diagonal [A] matrix being the easiest. The second term relates to the robustness of the design. The robustness is dictated by the [B] matrix that amplifies the errors δ**DP.**

4 TRANSFER FUNCTIONS OF A WATER FAUCET, AN EXAMPLE

We now illustrate the above mathematical treatment with an example. Consider the transfer functions of a water faucet. The two functional requirements are to control the water flow rate Q and to control its temperature T:

$$\mathbf{FR} = \begin{Bmatrix} FR_1 \\ FR_2 \end{Bmatrix} = \begin{Bmatrix} Q \\ T \end{Bmatrix}$$

Mass and energy conservation govern the mixing of hot and cold (subscripts *h, c*) water to yield a certain water flow rate Q and temperature T:

$$\text{mass conservation:} \quad Q = Q_h + Q_c \qquad (11)$$

$$\begin{aligned}\text{energy conservation:} \quad T &= \frac{Q_h T_h + Q_c T_c}{Q} \\ &= \frac{Q_h T_h + Q_c T_c}{Q_h + Q_c} \qquad (12a) \\ &= \left(\frac{Q_h/Q_c}{Q_h/Q_c + 1}\right)(T_h - T_c) + T_c \qquad (12b)\end{aligned}$$

To achieve the $\{Q,\ T\}$, we create a design that involves rotational angle θ of two valves that control separately the flow of hot and cold water. The design, a physical embodiment described in terms of the design parameters, is $Q_h = k_1\,\theta_1$, $Q_c =$

$k_I\ \theta_2$; with k_I being a proportional constant and $\{\theta_1, \theta_2\}$ the design parameters. With such a design, Equations (11) and (12a) are arranged into equations that map the input variables $\{\theta_1, \theta_2\}$ into the output quantities $\{Q, T\}$ as follows.

$$\begin{Bmatrix} Q \\ T \end{Bmatrix} = \begin{bmatrix} k_I & k_I \\ \dfrac{T_h}{\theta_1 + \theta_2} & \dfrac{T_c}{\theta_1 + \theta_2} \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}$$

The above matrix equation is the transfer functions of the faucet. Per Equation (5), the [A] matrix is:

$$[A] = \begin{bmatrix} k_I & k_I \\ \dfrac{T_h}{\theta_1 + \theta_2} & \dfrac{T_c}{\theta_1 + \theta_2} \end{bmatrix}$$

Every design has unique transfer functions. For example, an alternate faucet design is to use the vertical and horizontal angles of a lever to control the flow of water: $(Q_h+Q_c) = k_{II}\ \theta_v$, $Q_h / Q_c = k_{II}\ \theta_h$; k_{II} being a proportional constant. With this design, Equations (11) and (12b) are arranged in a different way, giving rise to strikingly different transfer functions:

$$\begin{Bmatrix} Q \\ T \end{Bmatrix} = \begin{bmatrix} k_{II} & 0 \\ 0 & \dfrac{T_h - T_c}{1 + \theta_h} \end{bmatrix} \begin{Bmatrix} \theta_v \\ \theta_h \end{Bmatrix} + \begin{Bmatrix} 0 \\ T_c \end{Bmatrix}$$

The [A] matrix in this case is diagonal:

$$[A] = \begin{bmatrix} k_{II} & 0 \\ 0 & \dfrac{T_h - T_c}{1 + \theta_h} \end{bmatrix}$$

The water faucet example reveals an insight into transfer functions. Although the governing physics in both designs are the same, different physical embodiments of design parameters bring about different transfer function architectures that affect in a different way, the ease of tuning for target and the robustness of a system. In other words, the transfer functions of a design is not determined by the governing physics but by the conception of the design and the physical embodiment of the concept.

5 A TRANSFER FUNCTION PERSPECTIVE OF AXIOMATIC DESIGN AND ROBUST DESIGN

Axiom I in Axiomatic Design deals with the independence of **FR**; and Axiom II; with information content necessary to satisfy the **FR**. In the past, the design matrix [DM] that relates the **FR** to the **DP** has been used to indicate both the independence and the associated information content. From a transfer function perspective as expressed by Equation (10), it is more logical to indicate independence and information content separately with matrices [A] and [B] respectively. A design with an [A] matrix that is diagonally dominant would indicate strong independence among the **FR** and therefore easy to tune. Since the system range of a design is directly proportional to the squared error of **FR**, then according to Equation (8), the [B] matrix is the logical indicator for information content. For the same amount of error **ΔDP**, a design whose [B] matrix is diagonally dominant and with smaller trace would have less information content to satisfy the **FR**.

A mentioned earlier, robust design exploits nonlinearity of the transfer functions to achieve a design on target at a reduced sensitivity. Therefore, implementation of robust design requires that the [A] matrix be a function of **DP** and that it be diagonally dominant. While it appears that [B] matrix is primarily responsible for the design sensitivity, it is in fact the [A] matrix that plays the major role. This is because the [B] matrix is derived from [A] matrix as follows. From Equation (9), an element of [B] matrix is:

$$\begin{aligned} b_{ij} &= \frac{\partial f_i}{\partial DP_j} \\ &= \frac{\partial\left(\sum_k a_{ik} DP_k\right)}{\partial DP_j} \\ &= a_{ij} + \sum_k \frac{\partial a_{ik}}{\partial DP_j} DP_k . \end{aligned}$$

If matrix [A] is a constant, then the [B] matrix is also a constant. In this case, there will be no opportunity to implement robust design. In other words, the conception of a design and the physical embodiment of the concept determine the [A] matrix and consequently the opportunity and ease for implementation of robust design.

6 REFERENCES

[1] Nam P. Suh, *Axiomatic Design*, Oxford University Press, 2001

[2] Taguchi, G., (1987) *Systems of Experimental Design,* Vol. 1 and 2. ASI press.