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A CONSTRAINT OPTIMIZATION PERSPECTIVE ON AXIOMATIC DESIGN

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ABSTRACT

This paper aims to provide a mathematical perspective for the two axioms in Axiomatic Design. Specifically, the Independence Axiom 1 and the Minimum Information Axiom 2 are viewed from the perspective of equality constraint optimization.

Axiomatic Design declares Axiom 1 and Axiom 2 to be axiomatic; that they cannot be proven nor derived from other principles or laws of nature. In fact, this paper shows that the concept and implementation of the two axioms parallel those of equality constraint optimization. The two axioms could have been derived from it.

This paper also shows that the qualifying condition imposed by Axiom 1 that the design matrix be triangular or diagonal is only a sufficient condition for functional independence. It is subset of a larger set that satisfies the necessary condition. Thus, the design that has been allowed by Axiom 1 and found by Axiom 2 to have the minimum information content may not necessarily be the design with minimum information content among the larger set.

Keywords: equality constraint optimization, functional independence, constraint qualification, Axiomatic Design.

1 INTRODUCTION

Axiomatic Design (AD) is a design framework built on two rules for mapping functional requirements (FRs) to design parameters (DPs). The two rules are assumed to be axiomatic. Namely, they are self-evident truths for which there are no counter-examples or exceptions. They cannot be proven nor derived from other laws or principles of nature, Suh [1990]. AD has been around for four decades already. Yet it has not caught 'fire' in design community. A principal reason is the axiomatic assumption AD imposed. It is difficult for designers to accept truth without proof. Some criticisms are: "AD people invoke axioms to avoid proof of theory" and "AD is not a mathematically valid method". The fact is logic and mathematical treatments have been provided to clarify and reinforce concepts in AD. For example, based on formal logic, Lu and Liu [2011] presented a theoretical underpinning to elucidate the delineation of "what" from "how", providing justification and execution of mapping and decomposition unique to AD. As another example, Rinderle [1982] developed the mathematics for measuring coupling: reangularity which measure how close a design matrix is to becoming a decoupled triangular matrix; and semangularity which measures how dominant the diagonal elements of a matrix is relative to its off-diagonal elements. It is a measure of how close the matrix is to becoming the uncoupled diagonal matrix. This paper is yet another effort to provide mathematical basis for AD.

The rest of this paper is organized as follows. In Section 2, we use an example involving single functional requirement to demonstrate the impact of constraint optimization on design. In Section 3 we develop the mathematical basis for constraint optimization involving multiple functional requirements. In Section 4 we view Axiom 1 and Axiom 2 in the context of the mathematical basis derived in Section 3. Concluding remarks then follow in Section 5.

2 CONSTRAINT OPTIMIZATION FOR SINGLE FUNCTIONAL REQUIREMEN - AN EXAMPLE

The power steering assembly in car consists of a vertical tubular "top hat" joined to a horizontal tubular housing (Figure 1). The steering valve rotates inside the top hat to direct fluid left/right for power steering. The top hat is made of cast iron (E=120,000MPa, μ=0.29) for wear resistance; the housing is made of aluminum (E=71,000MPa, μ=0.34) for weight reduction. Press fitting joints the two components of dissimilar material together. Figure 2 shows the cross-section of the assembly at the joint.

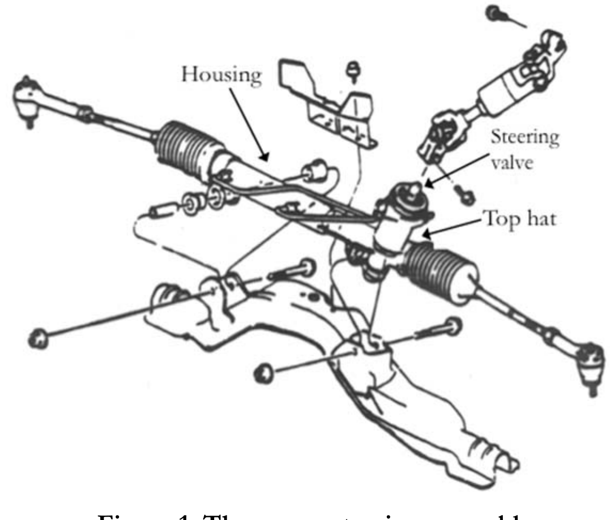

Figure 1. The power steering assembly.

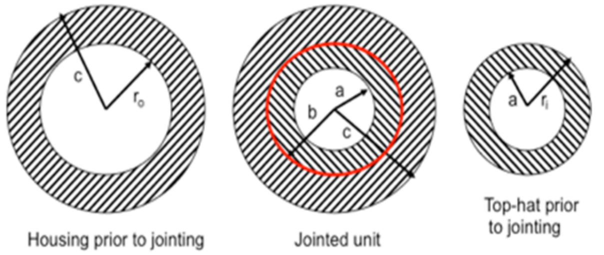

Figure 2. Cross-section of assembly at the joint.

One functional requirement FR is that the radial pressure developed at the interface holds the two components together. From an engineering handbook, the radial pressure p is given in Equation 1.

$$p = \frac{(r_i - r_o)}{\frac{b}{E_{AL}}\left(\frac{c^2+b^2}{c^2-b^2}+\mu_{AL}\right)+\frac{b}{E_{FE}}\left(\frac{b^2+a^2}{b^2-a^2}-\mu_{FE}\right)} \quad (1)$$

where $b = \left(\frac{r_i + r_o}{2}\right)$

so that $FR(DP_1, DP_2, DP_3, DP_4) = p = g(c, r_o, r_i, a)$,

where DP_1, DP_2, DP_3, DP_4 are respectively c, r_o, r_i, a.

To achieve a target value FR*, we solve Equation (1) for **DP*** that yields FR*. Hereafter bolded letters denote vectors. One approach is to minimize and reduce to zero the error:

$$\text{Error} = FR(DP_1, DP_2, DP_3, DP_4) - FR^* \quad (2)$$

The result is **DP*** = (26.00, 20.9738, 21.0161, 16.00) in millimeters which would give FR= 20 Pa, the target value. In our later discussion, we shall refer to this approach as nominal design.

In the presence of variability, FR will deviate from its target FR*. For example per Equation (1), a machining error of ±25μm in r_i and r_o will result in a radial pressure that ranges from -3.65Pa to 43.70Pa. (Axiomatic Design calls this range the system range.) At radial pressure < 0, solution by nominal design fails since a loose fit occurs at zero radial pressure.

The correct formulation is to pose the problem as an equality constraint optimization [Luenberger and Ye, 2008]. The designer should minimize the deviation due to variability, subject to the constraint that FR(***DP***) equals FR*, and thus expand FR(***DP***) in a Taylor series:

$$FR(DP) = FR^* + \sum_j \left.\frac{\partial FR}{\partial DP_j}\right|_{DP^*} \Delta NV_j \quad (3)$$

where *NV* denotes the noise variable, the source of variability; and the summation term is the deviation in *FR*. The *NV*s in this example are the radii r_i and r_o. So that:

$$FR(DP) = FR^* + \left.\frac{\partial FR}{\partial r_o}\right|_{DP^*} \Delta r_o + \left.\frac{\partial FR}{\partial r_i}\right|_{DP^*} \Delta r_i$$

Using squared deviation (SD) as the norm, we formulate the equality constraint optimization as follows:

$$\text{qualify } FR(DP): \quad \frac{\partial FR}{\partial DP_i} \neq 0; \quad \textit{for at least one } DP_i; \quad (4)$$

$$\text{minimize SD}: \quad \left(\left.\frac{\partial FR}{\partial r_o}\right|_{DP^*} \Delta r_o\right)^2 + \left(\left.\frac{\partial FR}{\partial r_i}\right|_{DP^*} \Delta r_i\right)^2; \quad (5)$$

$$\text{subject to}: \quad FR(DP) - FR^* = 0. \quad (6)$$

Qualification (4) is necessary. Otherwise, all partial derivatives of *FR*(***DP***) with respect to DP_i equal zero, *FR*(***DP***) will not be a function of ***DP***, and optimization cannot proceed. In our example, qualification (4) is satisfied because Equation (1) shows *FR*(***DP***) to be indeed a function of ***DP***. Expression (5) is the objective function to minimize. Equation (6) is the constraint equation that ***DP*** needs to satisfy at all times. In our discussion later, we shall call this approach of Equality Constraint Optimization the ECO design.

For both the nominal and ECO design, we use Excel to compute the sensitivity to variability and the squared deviation per Expression (5). The results, see Table 1 and Table 2, show that both ***DP**** (26.00, 20.9738, 21.0161, 16.00) from the nominal design and ***DP**** (25.00, 21.9363, 22.0000, 17.00) from the ECO design give *FR*=20Pa. However, sensitivity to variability is less with ECO design. Consequently, the squared deviation using the ECO design is only 36% that of the nominal design.

From this example, we conclude that we should adopt the ECO design and the equality constraint optimization approach.

Table 1. Sensitivy and squared deviation of nominal design.

Description	Nominal Value	Δr	Sensitivity ∂FR/∂r	Squared Deviation
Housing OR, c	26.00			
Housing IR, ro	20.9738	0.0250	-437.4396	119.5959
Top hat OR, ri	21.0161	0.0250	434.3347	117.9041
Top hat IR, a	16.00			
Radial Pressure	20.00		Total =	237.5000

Table 2. Sensitivity and squared deviation of ECO design.

Description	Nominal Value	Δr	Sensitivity ∂FR/∂r	Squared Deviation
Housing OR, c	25.00			
Housing IR, ro	21.9363	0.0250	-267.4571	44.7083
Top hat OR, ri	22.0000	0.0250	261.5885	42.7678
Top hat IR, a	17.00			
Radial Pressure	20.00		Total =	87.4762

3 CONSTRAINT OPTIMIZATION FOR MULTIPLE FUNCTIONAL REQUIREMENTS

For multiple functional requirements, ***FR***(***DP***) is a vector valued function of the form

$$FR(DP)=\begin{pmatrix} FR_1(DP) \\ FR_2(DP) \\ \vdots \\ FR_n(DP) \end{pmatrix}=\begin{pmatrix} FR_1(DP_1,DP_2,\cdots,DP_m) \\ FR_2(DP_1,DP_2,\cdots,DP_m) \\ \vdots \\ FR_n(DP_1,DP_2,\cdots,DP_m) \end{pmatrix}$$

We first qualify that ***FR***(***DP***)-***FR**** **= 0** is non-degenerate. Otherwise there can be no solution for ***DP*** and optimization cannot proceed. Given the system of equations:

$$\begin{pmatrix} FR_1(DP_1,DP_2,\cdots,DP_m) \\ FR_2(DP_1,DP_2,\cdots,DP_m) \\ \vdots \\ FR_n(DP_1,DP_2,\cdots,DP_m) \end{pmatrix}-\begin{pmatrix} FR_1^* \\ FR_2^* \\ \vdots \\ FR_n^* \end{pmatrix}=0$$

For above system of equations to be non-degenerate, m cannot be less than n. If m equals n, the determinant J of the Jacobian matrix of ***FR***(***DP***) must not be zero: $J \neq 0$. That is:

$$J=\begin{vmatrix} \frac{\partial FR_1}{\partial DP_1} & \frac{\partial FR_1}{\partial DP_2} & \cdots & \frac{\partial FR_1}{\partial DP_n} \\ \frac{\partial FR_2}{\partial DP_1} & \frac{\partial FR_2}{\partial DP_2} & \cdots & \frac{\partial FR_2}{\partial DP_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial FR_n}{\partial DP_1} & \frac{\partial FR_n}{\partial DP_2} & \cdots & \frac{\partial FR_n}{\partial DP_n} \end{vmatrix}\neq 0. \qquad (7)$$

If m is greater than n, then we choose n among the m *DP*s such that the associated Jacobian $J \neq 0$.

Note that the Jacobian matrix is, in fact, the design matrix [A] in Axiomatic Design:

$$[\mathrm{A}]\equiv\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}=\begin{bmatrix} \frac{\partial FR_1}{\partial DP_1} & \frac{\partial FR_1}{\partial DP_2} & \cdots & \frac{\partial FR_1}{\partial DP_n} \\ \frac{\partial FR_2}{\partial DP_1} & \frac{\partial FR_2}{\partial DP_2} & \cdots & \frac{\partial FR_2}{\partial DP_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial FR_n}{\partial DP_1} & \frac{\partial FR_n}{\partial DP_2} & \cdots & \frac{\partial FR_n}{\partial DP_n} \end{bmatrix}$$

Thus a re-statement of Equation (7) is that to qualify a system of equations ***FR***(***DP***)-***FR**** **= 0** for optimization, its Jacobian J, which is the determinant of [A] matrix, must not be zero:

$$J=|\mathrm{A}|=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}\neq 0. \qquad (8)$$

In testing for $J \neq 0$, we are in fact testing the functional independence of ***FR*** (***DP***) [Chiang, 1984].

To derive the expression for squared deviation, we first expand ***FR*** (***DP***) into n set of Taylor series:

$$FR(DP)=\begin{bmatrix} FR_1^* \\ FR_2^* \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ FR_n^* \end{bmatrix}+$$

$$+\left.\begin{bmatrix} \frac{\partial FR_1}{\partial NV_1} & \frac{\partial FR_1}{\partial NV_2} & \cdots & \frac{\partial FR_1}{\partial NV_l} \\ \frac{\partial FR_2}{\partial NV_1} & \frac{\partial FR_2}{\partial NV_2} & \cdots & \frac{\partial FR_2}{\partial NV_l} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial FR_n}{\partial NV_1} & \frac{\partial FR_n}{\partial NV_2} & \cdots & \frac{\partial FR_n}{\partial NV_l} \end{bmatrix}\right|_{DP^*}\begin{bmatrix} \Delta NV_1 \\ \Delta NV_2 \\ \vdots \\ \vdots \\ \vdots \\ \Delta NV_l \end{bmatrix}$$

Each ith equation above is a Taylor series expansion of $FR_i(DP)$ similar to Equation (3). The above equation may be written in matrix form:

$$FR(DP)=\{FR^*\}+[\ B\ ]\big|_{DP^*}\{\Delta NV\}$$

where the [B] matrix is the Jacobian matrix of ***FR*** with respect to the noise variable ***NV*** with element

$$b_{ik}\equiv\frac{\partial FR_i}{\partial NV_k}, \quad i=1,2,\cdots,n; \quad k=1,2,\cdots,l.$$

The squared deviation (SD) is then the inner product:

$$SD=\{\Delta NV\}^T[B]^T[B]\{\Delta NV\}$$

The formulation for equality constraint optimization of multiple functional requirements is an extension of Equations (4), (5) and (6) as follows:

$$\text{qualify } FR(DP): \quad J = |A| \neq 0. \tag{9}$$

$$\text{minimize SD}: \quad [B]^T [B] \tag{10}$$

$$\text{subject to}: \quad FR(DP) - FR^* = 0 \tag{11}$$

4 AXIOMATIC DESIGN IN THE CONTEXT OF EQUALITY CONSTRAINT OPTIMIZATION

Axiomatic Design (AD) is built on two axioms [Suh, 1990]. Axiom 1 is a rule that qualifies a design as acceptable only if its FRs maintains independence. Of those that qualify, Axiom 2 then selects the one that has minimum information content. The process behind the two axioms, qualifying designs for functional independence followed by searching among the qualified designs for one with minimum information content, is similar to the formulation of equality constraint optimization. We therefore view AD in that light.

4.1 INDEPENDENCE AXIOM 1

According to Equation (8), a constraint qualification for **FR(*DP*)** is that its Jacobian J, i.e., the determinant of [A] matrix, not be zero. This is a necessary condition N.

In AD, Axiom 1 requires the design matrix [A] to be either diagonal or triangular. Since the determinant of these two types of matrices is not zero, the Axiom 1 requirement does fulfill the constraint qualification imposed by Equation (8). This also means that the FRs so qualified are functionally independent.

However, the condition that [A] be diagonal or triangular is only a sufficient condition S for $|A| \neq 0$. It is a subset of the larger set N that satisfies the necessary condition (Figure 3) Therefore, there can be designs whose design matrix [A] is neither diagonal nor triangular and yet its determinant J $\neq$ zero. These designs continue to be functionally independent. They may possess information content lower than the minimum found among the subset S. Thus in using Axiom 1 to qualify design, AD may completely miss these designs.

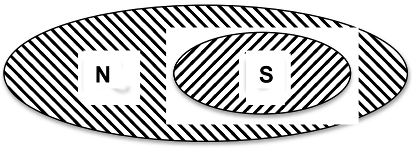

Figure 3. Sufficient condition as a subset of necessary condition.

4.2 INFORMATION AXIOM 2

Both ECO design and AD acknowledge the presence of variability and the associated uncertainty in design. Both use deviation in *FR* from the target as the metric for variability. ECO design uses squared loss to quantify loss due to deviation: the farther the deviation from the target, the larger the loss (Figure 4). It delves deeper to identify the sources of the variability **NV**, and compute the matrix [B], the sensitivity of *FR* to these sources. With $[B]^T[B]$ as the objective function, it becomes possible to minimize sensitivity for reduced deviation.

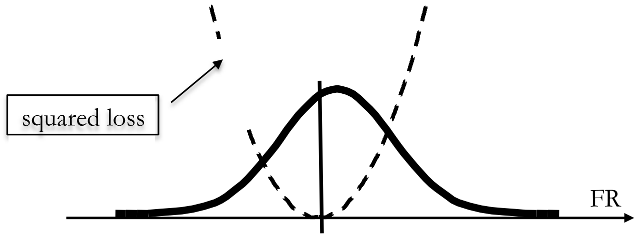

Figure 4. Squared loss function.

AD measures variability in terms of the range of deviation and calls it the system range. It then uses absolute loss to quantify the loss due to deviation. Absolute loss defines a range in *FR*, known as design range, center on the target value *FR** (Figure 5). A design whose deviation in *FR* falls within the design range incurs no loss. Otherwise, it will incur a loss of $(1-p)$, where p is given by:

$$p = \frac{\text{common range}}{\text{system range}}$$

The common range is the overlap of the design range and system range shown in Figure 5.

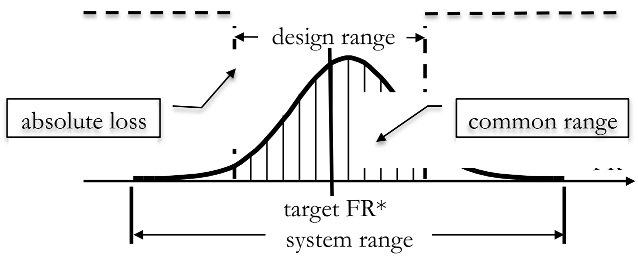

Figure 5. Design range, common range & system range.

AD further defines a quantity called the information content *I* as:

$$I = -\log_2\left(\frac{\text{common range}}{\text{system range}}\right).$$

Axiom 2 then uses information content *I* as the metric to select the design with the least information content *I* from among the designs qualified by Axiom 1.

Since AD adopts an absolute loss function, designs like A and B in Figure 6 whose system range fall within the design range are deemed equally good. Both have zero information. Thus it is equally likely that Axiom 2 will pick A over B or B over A as the best design. This is counter-intuitive. Intuition tells us that design B is the better because it has a larger margin for error.

Unlike ECO design, AD does not attempt to identify sources of variability nor provide an objective function to minimize. Its treatment of uncertainty in design is less extensive than that of the ECO design.

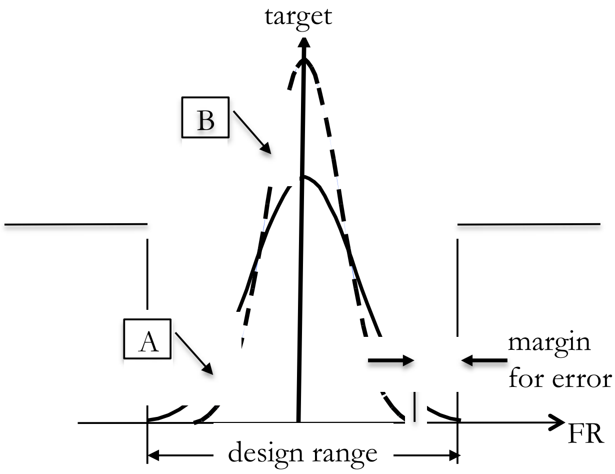

Figure 6. Design A versus Design B.

5 CONCLUDING REMARKS

AD declares Independence Axiom 1 and Minimum Information Axiom 2 to be axiomatic; that they cannot be proven nor derived from other principles or laws of nature. We have shown, in fact, that the concept and implementation of AD, i.e., qualifying design for functional independence followed by searching for the one design with minimum uncertainty, parallel those found in the decades-old equality constraint optimization. The concept and approach in AD could have been derived from it. Hence, there is no need to invoke axiomatic assumptions about them.

The qualifying condition imposed by Axiom 1 that design matrix [A] be triangular or diagonal is only a sufficient condition S for functional independence. It is subset of the larger set that satisfies the necessary condition N. Thus, the design that has been allowed by Axiom 1 and found by Axiom 2 to have the minimum information content may not necessarily be the design with minimum information content among the N set. If a design outside the S set is found to have lower information content, then a counter example exists; and Axiom 1 and 2 do not hold.

In adopting an absolute loss function, Axiom 2 at times produces conclusions that are counter-intuitive. It is suggested that square loss function be used instead.

AD involvement in assessing uncertainty in design should be taken to a larger extend than it currently is. AD should begin to recognize and search for the sources of variability ***NV***, sensitivity of ***FR*** to them, and try to reduce the sensitivity to achieve a reduced loss.

AD offers many other concepts and approaches: top down zigzag decomposition of FR-DP; separation of domains to provide a neutral environment for defining FRs; an environment conducive to bi-modal, linear and non-linear, thinking, etc. These are all unique to AD. Hence the name Axiomatic Design should be kept even though there is no need to invoke axiomatic assumption of the method.

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