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New Formulations of Design Optimization for Six-sigma, Reliability and Robustness

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Contents

Optimal Design Problem Formulation

- Elements of an optimal design problem
  - Objective function (Single or multi-objective)
  - Design variables (Size, shape, topology, concept)
  - Constraints
  - System equations
  - Systems with uncertainty →Robust design, RBDO
- Information on probability distributions of uncertain variables necessary

Issues for RBDO and Kwak's contributions

- How to calculate Reliability
  - Monte Carlo simulation, Reliability index,…
  - Moment with DOE, Expanding Response Surface Moment Method (RSMM)
- How to treat probability constraints --Sub-optimization problem
  - RIA (Reliability index approach)
  - FNA (Fixed norm approach)
- How to solve the RBDO
  - Approximate gradient approach: FDM, Utilize DOE
  - Gradient-based approach: Sensitivity using POD
- DOE based RBDO Procedure

(Moment using 3 pt info) + (Pearson sys) + (Sensitivity with DOE)

Issues for Robust Optimal Design

- How to define Robust Optimal Design?
  - Minimum "sensitivity" to "uncertainties"
  - → No universal consensus yet
- How to formulate a robust optimal design?
  - Using RBDO → Too expensive to get solution and probability information
  - Without using probability info → Simple and efficient
    - Using GI
    - Allowable set approach (ALS)
    - Taguchi method

Robust optimal design with Gradient Index

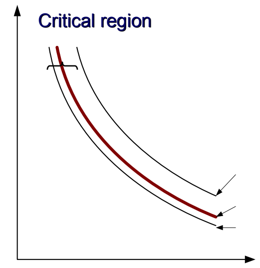
- Conventional optimization formulation

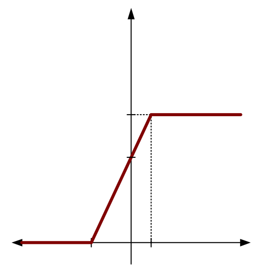
$$\begin{aligned} &\text{Minimize} && f(\mathbf{x},\mathbf{z}) \\ &\text{subject to} && g_j(\mathbf{x},\mathbf{z}) \le 0 \quad j=1,2,\ldots,m \\ & && \mathbf{x}_L \le \mathbf{x} \le \mathbf{x}_U \end{aligned}$$

- Robust optimization formulation using GI

$$\begin{aligned} &\text{Minimize} && \text{GI} \\ &\text{subject to} && g_j(\mathbf{x},\mathbf{z}) + \Psi_j(g_j(\mathbf{x},\mathbf{z})) \le 0 \quad j=1,2,\ldots,m \\ & && f(\mathbf{x},\mathbf{z}) \cong M \end{aligned}$$

Feasibility robustness using GI_{gj}





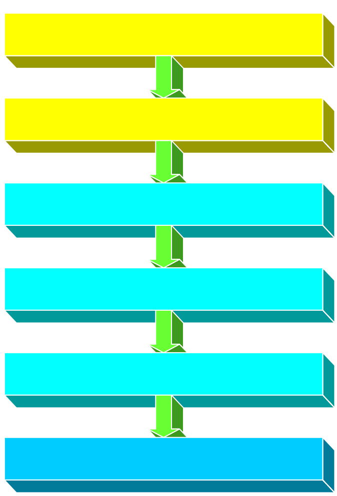
$$\Psi_j(g_j) = \begin{cases} 0 & g_j < CT \\ \dfrac{\kappa_j GI_{gj}}{CTMIN - CT}(g_j - CT) & CT \le g_j \le CTMIN \\ \kappa_j GI_{gj} & g_j > CTMIN \end{cases}$$

CT: small negative value
CTMIN: small positive value
κ_j: factor for GI_{gj}

$$GI_{gj} = \max_i \left| \frac{dg_j}{du_i} \right| \quad i = 1,2,\ldots,N$$

Robust optimal design procedure for MEMS structures

X_2



- Identify important characteristics and constraints
- Find out what we can modify or change

- Perform deterministic optimizations to improve the performances

- Select uncertain variables in consideration of fabrication processes and other uncertainties

- SA for objective and constraint functions w.r.t. u_i
- For both initial and deterministic optimal designs

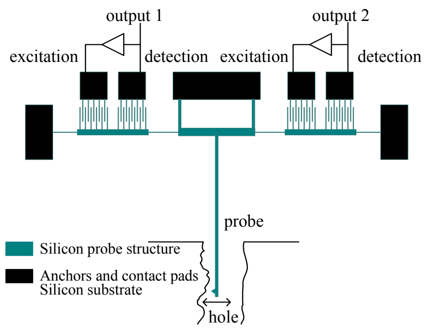
- Select critical u_i and responses for robust design
- Define GI

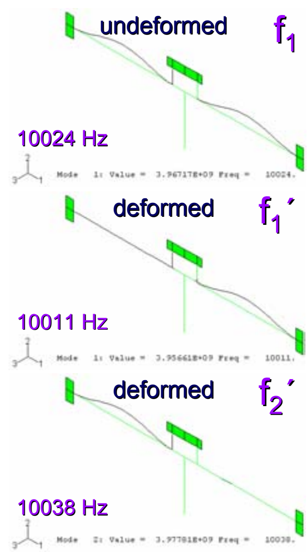
- Perform robust optimizations using the GI

Ex1: Resonant-type micro probe

[Lebrasseur, 2000]

Principle of operation



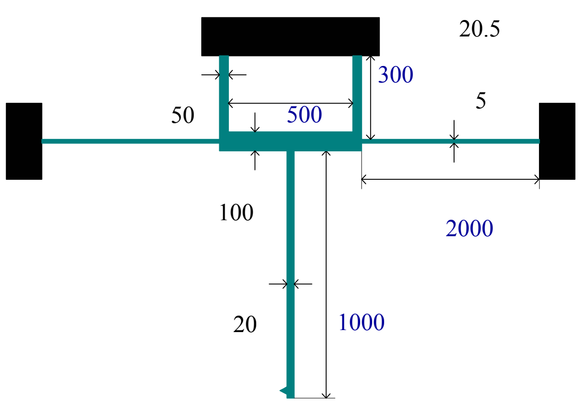


- High aspect ratio micro holes
- Stress-induced frequency shift
- Working mode is the first resonant mode

Measurement sensitivity: $\Delta f / f\ (\%) = (f_2' - f_1') / f_1 \times 100$

Identification

Design variables



$1000 \le x_1 \le 1500$

$1000 \le x_2 \le 3000$

$200 \le x_3 \le 500$

$300 \le x_4 \le 800$

$3 \le x_5 \le 80$

$3 \le x_6 \le 30$

$10 \le x_7 \le 100$

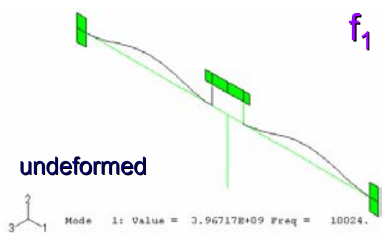
$20 \le x_8 \le 200$

$15 \le x_9 \le 25$

Unit: μm

Deterministic optimization

$$
\begin{aligned}
&\text{Maximize} && \Delta f / f\ (\%) && \text{: measurement sensitivity} \\
&\text{subject to} && f_1 \geq 10000\,\text{Hz} && \text{: } freq_1 \geq 10\text{ kHz} \\
& && f_3 \geq 20000\,\text{Hz} && \text{: } freq_3 \geq 20\text{ kHz} \\
& && \sigma_{max} \leq \sigma_{yield} && \text{: } \sigma_{max(1\ \mu m)} \leq \sigma_{yield} \\
& && f_{\Gamma 1i} - f_1 \leq 0.001 && \text{: tracing the mode shape} \\
& && \Delta f / f \cong M && \text{: } M^{\dagger} = 6\ \%
\end{aligned}
$$

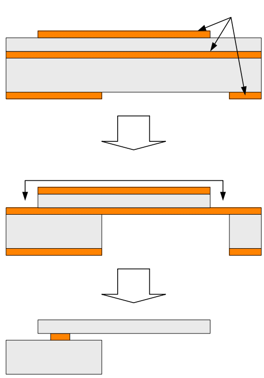


$$
\Gamma_{1i} = \begin{Bmatrix} 0 \\ 1.3 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}
$$

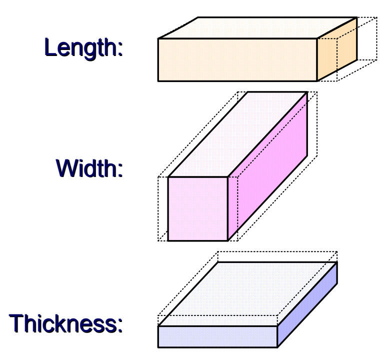
†: The sensitivity was 2~3 % in the reference [15]

Selection of uncertain variables

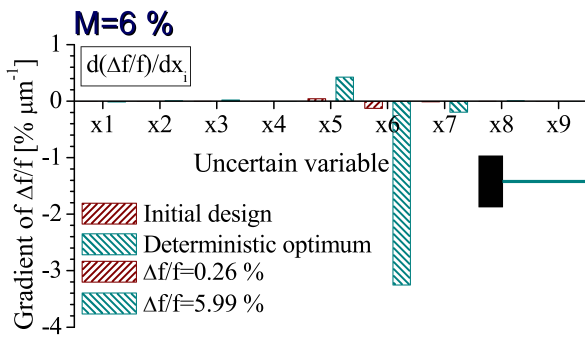
- Fabrication process

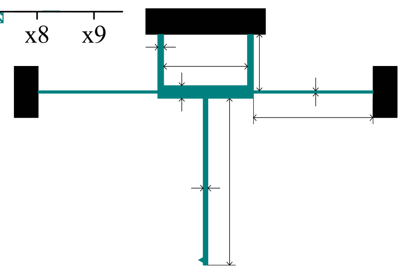


- Error patterns



Sensitivity of $\Delta f/f$ w.r.t. uncertain variables





⇒ **Uncertain variables:** $u_i = \{x_5, x_6, x_7, x_8, x_9\}$

Robust optimization

Deterministic optimization

Maximize	$\Delta f / f$ (%)
subject to	$f_1 \geq 10000\,\text{Hz}$
	$f_3 \geq 20000\,\text{Hz}$
	$\sigma_{max} \leq \sigma_{yield}$
	$f_{\Gamma 1i} - f_1 \leq 0.001$
	$\Delta f / f \cong M$

Robust optimization: M=6 %

Minimize	GI
subject to	$g_j + \Psi_j(g_j) \leq 0 \quad j = 1,2,3$
	$f_{\Gamma 1i} - f_1 \leq 0.001$
	$\Delta f / f \cong M$

$$GI = \max_k \left| \frac{d(\Delta f / f)}{du_k} \right|$$

$$u_i = \{x_5, x_6, x_7, x_8, x_9\}$$

Results

Design variable		Lower bound	Initial design	M=6 % Deterministic optimum	Robust optimum 1†	Robust optimum 2‡	Upper bound
x_1	μm	1000	1000	1000.0	1000.0	1000.0	1500
x_2	μm	1000	2000	1991.4	2004.7	1997.9	3000
x_3	μm	200	300	300.9	260.2	248.1	500
x_4	μm	300	500	497.2	478.8	483.0	800
x_5	μm	3	20	42.2	79.9	79.9	80
x_6	μm	3	5	4.9	7.9	7.6	30
x_7	μm	10	50	31.6	47.8	46.6	100
x_8	μm	20	100	115.1	181.8	168.7	200
x_9	μm	15	20.5	24.9	25.0	25.0	25
$\Delta f/f$	%		0.26	5.99	5.99	5.99	
GI	% μm^{-1}		0.12	3.25	1.81	1.84	
f_1	Hz		10024	9999	15768	15439	
f'_1	Hz		10011	9695	15287	14968	
f'_2	Hz		10037	10295	16233	15894	

†Robust optimum 1 : No feasibility robustness

‡Robust optimum 2 : Feasibility robustness by Ψ_j with κ_1=2.0, κ_2=2.0, κ_3=2.0

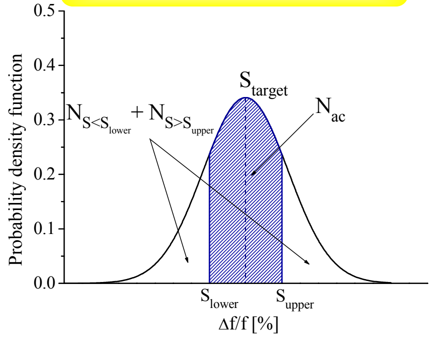
Monte Carlo simulation

M=6 %			Deterministic optimum	Robust optimum 1	Robust optimum 2
			Variation II‡	Variation II	Variation II
Mean ± Standard deviation	$\Delta f/f$	%	5.99 ± 1.14	5.99 ± 0.62	5.99 ± 0.63
	f_1	Hz	10000 ± 669	15768 ± 660	15439 ± 665
	f_3	Hz	20000 ± 149	19980 ± 138	20146 ± 139
	σ_{max}	MPa	8.7 ± 0.06	16.4 ± 0.06	16.3 ± 0.06
Violation probability	g_1	%	50.0	0.0	0.0
	g_2	%	41.7	46.7	10.6
	g_3	%	0.0	0.0	0.0
	g_4	%	0.0	0.0	0.0
Yield	SY	%	62.4	90.0	89.7

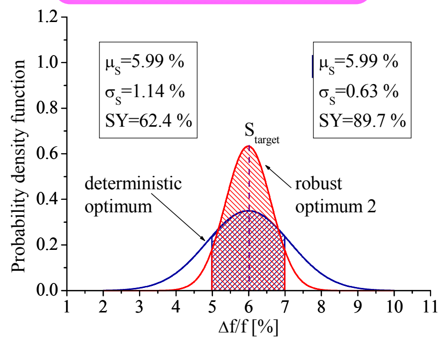
‡Variation II : $\Delta x_1=\Delta x_2=\Delta x_3=\Delta x_4=\pm 2.0$ μm, $\Delta x_5=\Delta x_6=\Delta x_7=\Delta x_8=\pm 1.0$ μm and $\Delta x_9=\pm 0.5$ μm

Yield†

$$SY(\%) = \frac{N-(N_{g4}+N_S)}{N} \times 100(\%)$$



$\Delta x_1=\Delta x_2=\Delta x_3=\Delta x_4=\pm 2.0\ \mu m$
$\Delta x_5=\Delta x_6=\Delta x_7=\Delta x_8=\pm 1.0\ \mu m$
$\Delta x_9=\pm 0.5\ \mu m$



†: Monte Carlo simulation using 1000 samples

Structural Reliability

Reliability

-Ability to fulfill the design purpose for an intended period

-- Probability that a structure will not reach any specified limit state

Limit State

State beyond which a structure, or a part of it, can no longer fulfill the functions or satisfy the conditions for which it was designed

E.g.: Stress condition, Displacement condition, Frequency
Buckling conditions, etc.

Failure Function / Limit State Function $G(\mathbf{X})$

The boundary of a safety domain D in the space of random variable $\mathbf{X}$

Optimal design formulation--probabilistic

$$\min \quad W(\mathbf{b})$$

$$H_i(\mathbf{b},\mathbf{z},\mathbf{x}) = 0 \quad i = 1,\cdots,s$$ (State eqn)

$$\Pr\left[\bigcup_{j=1}^{m}\{G_j(\mathrm{b},\mathrm{z},\mathrm{x}) \le 0\}\right] \le p_0$$ (Sys failure mode)

$$\Pr\left[G_j(\mathrm{b},\mathrm{z},\mathrm{x}) \le 0\right] \le p_j, \ j = m+1,\ldots,m'$$

$$G_j(\mathrm{b}) \le 0, \quad j = m'+1,\ldots,m''$$

Reliability: $1 - P_f$

$$P_f = \Pr\left[G(\mathbf{X}) \le 0\right] = \int_{G(\mathbf{X})\le 0} f(\mathbf{X})\,d\mathbf{X}$$

Reliability analysis: Level 3 Methods

Monte Carlo Simulation

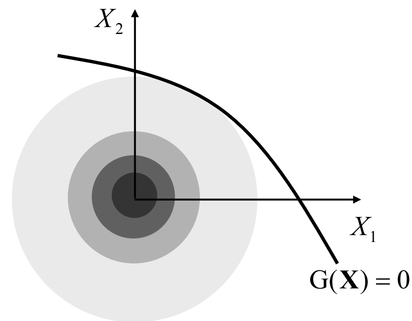
Concept

$$P_f = \Pr[G(\mathbf{X}) \le 0] = \int_{G(\mathbf{X})\le 0} f(\mathbf{X})d\mathbf{X}$$

$$P_f = \int I[G(\mathbf{X}) \le 0] f(\mathbf{X})d\mathbf{X}$$

$$P_f \approx \frac{1}{N}\sum_{i=1}^{N} I[G(\mathbf{X}_i) \le 0]$$

$I[\bullet]$: Indicator function



1. Generate random samples for X_i
2. Compute $G(\mathbf{X}_i)$ and $I[G(\mathbf{X}_i) \le 0]$
3. Calculate P_f

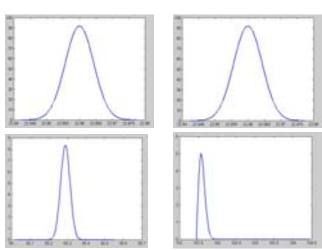
- Too much calculation and accuracy problem

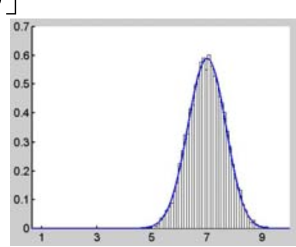
Moment method

To find probability moments and eventually the distribution, y.

Uncertainties $\mathbf{x}$ → **ENGINEERING SYSTEM**: Fortini's Clutch → **Performance** $y < 9^\circ$

$$y = g(\mathbf{x}) = \arccos\left[\frac{x_1 + \frac{1}{2}(x_2 + x_3)}{x_4 - \frac{1}{2}(x_2 + x_3)}\right]$$



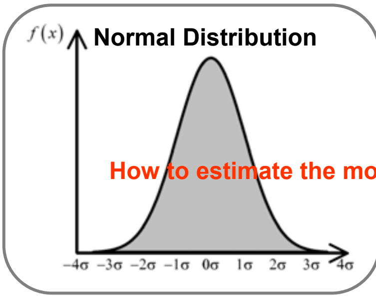


Design of Experiment

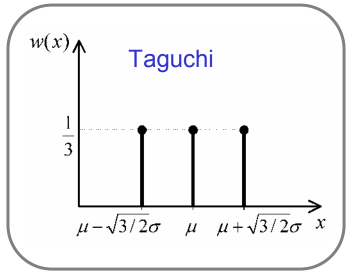
Three-Level Taguchi Method

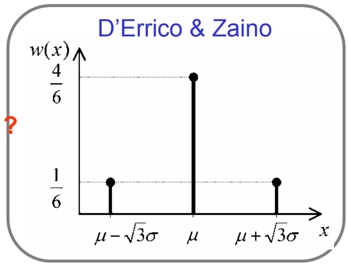
$$E\{g^k\} = \int_{-\infty}^{+\infty} [g(x)]^k \phi\left(\frac{x-\mu}{\sigma}\right) dx$$

$$\cong \sum_{i=1}^{m} w_i [g(\mu + \alpha_i \sigma)]^k$$



How to estimate the moments of g ?





Full Factorial DOE

3^n Factorial Design : Normal Distribution

$$E\{g^k\} = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} [g(x_1, \cdots, x_n)]^k \prod_{i=1}^{n} \phi\left(\frac{x_i - \mu_i}{\sigma_i}\right) dx_1 \cdots dx_n$$

$$\cong \sum_{j_1=1}^{m} w_{j_1} \cdots \sum_{j_n=1}^{m} w_{j_n} \left[g(\mu_1 + \alpha_{j_1}\sigma_1, \cdots, \mu_n + \alpha_{j_n}\sigma_n)\right]^k$$

(D'Errico & Zaino)

Exp.	Level		Weight
k	j_1	j_2	$W(k)$
1	1	1	1/36
2	1	2	4/36
3	1	3	1/36
4	2	1	4/36
5	2	2	16/36
6	2	3	4/36
7	3	1	1/36
8	3	2	4/36
9	3	3	1/36

$y = g(x_1, x_2, \ldots, x_n)$ → $g(\mathbf{x}(k))$, $W(k) = w_{j_1} w_{j_2} \cdots w_{j_n}$

1. mean

$$\mu_g = \sum_{k=1}^{3^n} g(\mathbf{x}(k)) W(k)$$

2. standard deviation

$$\sigma_g = \sqrt{\sum_{k=1}^{3^n} \left[g(\mathbf{x}(k)) - \mu_g\right]^2 W(k)}$$

3. skewness

$$\sqrt{\beta_{1g}} = \sum_{k=1}^{3^n} \frac{\left[g(\mathbf{x}(k)) - \mu_g\right]^3}{\sigma_g^3} W(k)$$

4. kurtosis

$$\beta_{2g} = \sum_{k=1}^{3^n} \frac{\left[g(\mathbf{x}(k)) - \mu_g\right]^4}{\sigma_g^4} W(k)$$

DOE for non-normal distributions

3^n Factorial Design : Non-normal Distribution

$$M_k = \int_{-\infty}^{+\infty} (x-\mu)^k f(x)dx$$
$$\cong w_1 (l_1-\mu)^k + w_2 (l_2-\mu)^k + \cdots + w_m (l_m-\mu)^k \qquad (k = 0,1,2,3,4, and\ 5)$$

$$w_1 + w_2 + w_3 = 1$$
$$w_1 l_1 + w_2 l_2 + w_3 l_3 = \mu$$
$$(l_1-\mu)^2 w_1 + (l_2-\mu)^2 w_2 + (l_3-\mu)^2 w_3 = \sigma^2$$
$$\frac{(l_1-\mu)^3 w_1 + (l_2-\mu)^3 w_2 + (l_3-\mu)^3 w_3}{\sigma^3} = \sqrt{\beta_1}$$
$$\frac{(l_1-\mu)^4 w_1 + (l_2-\mu)^4 w_2 + (l_3-\mu)^4 w_3}{\sigma^4} = \beta_2$$
$$(l_1-\mu)^5 w_1 + (l_2-\mu)^5 w_2 + (l_3-\mu)^5 w_3 = M_5$$

(Seo & Kwak 2002)

$$l_2 = \mu + \Delta$$

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Design of Experiments

Set $\Delta = 0$

$$\{l_1, l_2, l_3\} = \begin{bmatrix} \mu + \frac{\sqrt{\beta_1}\,\sigma}{2} - \frac{\sigma}{2}\sqrt{4\beta_2 - 3\beta_1} \\ \mu \\ \mu + \frac{\sqrt{\beta_1}\,\sigma}{2} + \frac{\sigma}{2}\sqrt{4\beta_2 - 3\beta_1} \end{bmatrix}^T, \quad \{w_1, w_2, w_3\} = \begin{bmatrix} \frac{(4\beta_2 - 3\beta_1) + \sqrt{\beta_1}\sqrt{4\beta_2 - 3\beta_1}}{2(4\beta_2 - 3\beta_1)(\beta_2 - \beta_1)} \\ \frac{\beta_2 - \beta_1 - 1}{\beta_2 - \beta_1} \\ \frac{(4\beta_2 - 3\beta_1) - \sqrt{\beta_1}\sqrt{4\beta_2 - 3\beta_1}}{2(4\beta_2 - 3\beta_1)(\beta_2 - \beta_1)} \end{bmatrix}^T$$

Statistical moments of $g(x_1, x_2, \ldots, x_n)$: Product Quadrature Rule

$$\mu_g = \sum_{i_1=1}^{3} w_{i_1} \cdots \sum_{i_n=1}^{3} w_{i_n} g(l_{i_1}, \cdots, l_{i_n}), \qquad \sigma_g = \left[\sum_{i_1=1}^{3} w_{i_1} \cdots \sum_{i_n=1}^{3} w_{i_n} \left(g(l_{i_1}, \cdots, l_{i_n}) - \mu_g \right)^2 \right]^{1/2},$$

$$\sqrt{\beta_1}_g = \left[\sum_{i_1=1}^{3} w_{i_1} \cdots \sum_{i_n=1}^{3} w_{i_n} \left(g(l_{i_1}, \cdots, l_{i_n}) - \mu_g \right)^3 \right] \Big/ \sigma_g^{\,3}, \quad \beta_{2g} = \left[\sum_{i_1=1}^{3} w_{i_1} \cdots \sum_{i_n=1}^{3} w_{i_n} \left(g(l_{i_1}, \cdots, l_{i_n}) - \mu_g \right)^4 \right] \Big/ \sigma_g^{\,4}.$$

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Full factorial DOE procedure (FFMM)

Reliability Analysis Using DOE (Seo & Kwak 2002)

Given Component Distribution or Raw Data

↓ *Calculate the first four moments of* x_i

$\mu_{x_i}, \sigma_{x_i}, \sqrt{\beta_1}_{x_i}$ and β_{2x_i}

↓ *Determine levels and weights*

$\{l_{1_i}, l_{2_i}, l_{3_i}\}$ and $\{w_{1_i}, w_{2_i}, w_{3_i}\}$

↓ *Run Design of Experiments*

$\mu_g, \sigma_g, \sqrt{\beta_1}_g$ and β_{2g}

↓ *Compute Probability using Pearson System*

Probability of Failure : $\Pr[g(\mathbf{x}) < 0]$

Probability distribution system

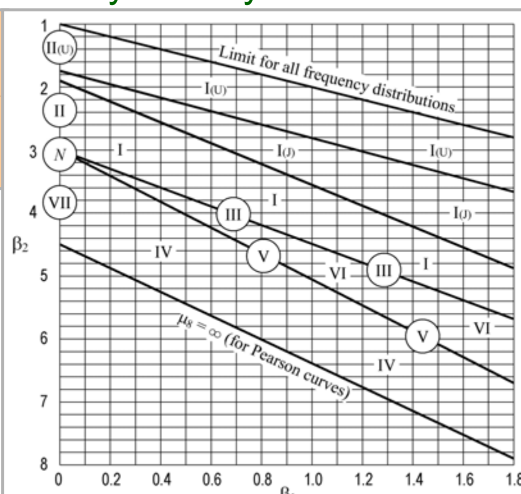
Pearson System – Probability Density Function

Pearson system

$$\frac{d}{dX}\left(\log f(x)\right) = \frac{X}{B_0 + B_1 X + B_2 X^2}$$

$$\text{where} \quad X = x - \mu$$

Type I	beta distribution
Type II	symmetrical form of the function defined in Type I
Type III	gamma distribution
Type IV	no common statistical distributions are of the type
Type V	inverse Gaussian distribution
Type VI	cumulative Pareto distribution
TypeVII	t distribution

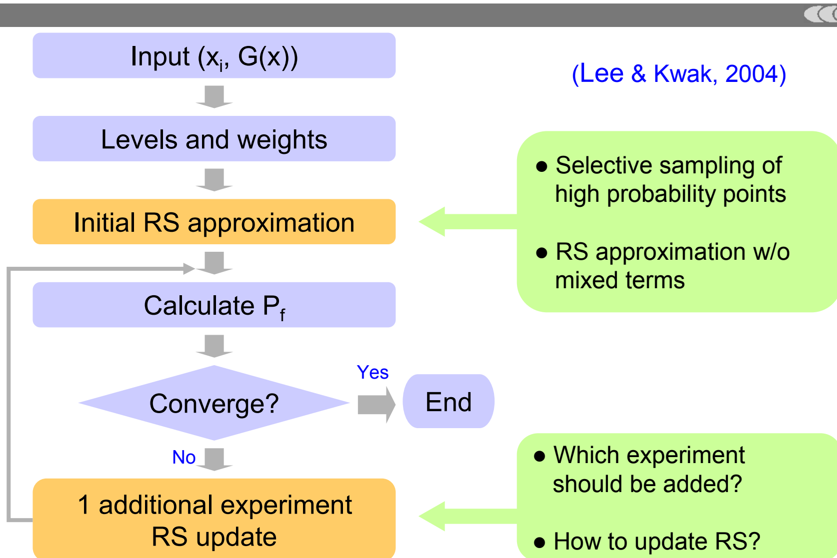


Response Surface-Moment Method (RSMM)

- Motivation
  - Full factorial moment method (FFMM) is too expensive to use. # of uncertain parameters < 10
- Strategy
  1. Strong points of FFMM or 3^n DOE must be preserved. → Utilize the levels and weights by Seo & Kwak.
  2. Retrieve as much information as possible from experimental data. → Utilize the same data for moment estimation and curve fitting.
  3. The number of experiments is increased adaptively to obtain the accuracy of FFMM.

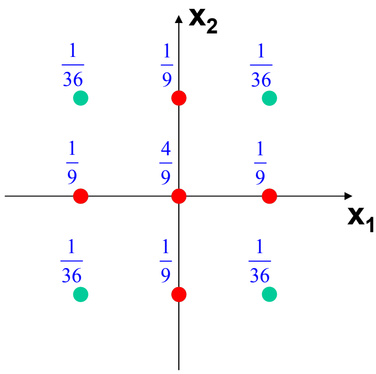
     → Selective sampling and RS update utilized.

Expanding RS-Moment Method (RSMM)



Initial approximation

Experiments with high probability concentration



$$w_i = w_{1i} \bullet w_{2i} \bullet \cdots \bullet w_{ni} = \prod_{j=1}^{n} w_{ji}$$

w_{ji} : weight of j-th variable at i-th experiment (j=1~n, i=1~3^n)

$$\tilde{G}(\mathbf{x}) = a + \sum_{i=1}^{n} b_i x_i + \sum_{i=1}^{n} c_i x_i^2$$

- Total **(1+2n)** REAL experiments
- $G(\mathbf{x})$ is approximated by $\tilde{G}(\mathbf{x})$ for the rest of experimental points

Least square approximation

$$y_i = a_0 + b_1 x_{i1} + \cdots + b_n x_{in} + c_1 x_{i1}^2 + \cdots c_n x_{in}^2 + \varepsilon_i$$
$$= a_0 + \sum_{j=1}^{n} b_j x_{ij} + \sum_{j=1}^{n} c_j x_{ij}^2 + \varepsilon_i$$

$$\mathbf{y} = \mathbf{X\beta} + \boldsymbol{\varepsilon}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} 1 & x_{11} & \cdots & x_{1n} & x_{11}^2 & \cdots & x_{1n}^2 \\ 1 & x_{21} & \cdots & x_{2n} & x_{21}^2 & \cdots & x_{2n}^2 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 1 & x_{k1} & \cdots & x_{kn} & x_{k1}^2 & \cdots & x_{kn}^2 \end{bmatrix}$$

$$\boldsymbol{\beta} = \begin{bmatrix} a_0 \\ b_1 \\ \vdots \\ c_n \end{bmatrix}, \quad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_k \end{bmatrix}$$

$$L = \sum_{i=1}^{k} \varepsilon_i^2 = \boldsymbol{\varepsilon}^T \boldsymbol{\varepsilon} = (\mathbf{y} - \mathbf{X\beta})^T (\mathbf{y} - \mathbf{X\beta})$$

$$L = \mathbf{y}^T\mathbf{y} - \boldsymbol{\beta}^T\mathbf{X}^T\mathbf{y} - \mathbf{y}^T\mathbf{X\beta} + \boldsymbol{\beta}^T\mathbf{X}^T\mathbf{X\beta}$$
$$= \mathbf{y}^T\mathbf{y} - 2\boldsymbol{\beta}^T\mathbf{X}^T\mathbf{y} + \boldsymbol{\beta}^T\mathbf{X}^T\mathbf{X\beta}$$

$$\left.\frac{\partial L}{\partial \boldsymbol{\beta}}\right|_{\mathbf{b}} = -2\mathbf{X}^T\mathbf{y} + 2\mathbf{X}^T\mathbf{X}\mathbf{b} = 0$$

$$\mathbf{b} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}$$

$\mathbf{b}$: Least square estimator of $\boldsymbol{\beta}$

Selection of new experimental point

- Find a candidate point which is likely to make the biggest change in Pf.

$$\Delta P_f = \frac{dP_f}{d\tilde{g}_i}(\tilde{g}_i - g_i)$$

  - g_i : value of $G(\mathbf{x})$ at i-th experimental point (i=1~3^n-2n-1)
  - $\tilde{g}_i$: value of $\tilde{G}(\mathbf{x})$ at i-th experimental point (i=1~3^n-2n-1)

- $\left|\frac{dP_f}{d\tilde{g}_i}\right|$ is selected as a measure of the relative importance of experimental points, to be called influence index

Influence Index

- Calculation of Influence Index

$$P_f = P_f(\mu, \sigma, \sqrt{\beta_1}, \beta_2)$$

$$\frac{dP_f}{dg_i} = \frac{\partial P_f}{\partial \mu}\cdot\frac{\partial \mu}{\partial g_i} + \frac{\partial P_f}{\partial \sigma}\cdot\frac{\partial \sigma}{\partial g_i} + \frac{\partial P_f}{\partial \sqrt{\beta_1}}\cdot\frac{\partial \sqrt{\beta_1}}{\partial g_i} + \frac{\partial P_f}{\partial \beta_2}\cdot\frac{\partial \beta_2}{\partial g_i}$$

$$\approx \frac{\Delta P_f}{\Delta \mu}\cdot\frac{\partial \mu}{\partial g_i} + \frac{\Delta P_f}{\Delta \sigma}\cdot\frac{\partial \sigma}{\partial g_i} + \frac{\Delta P_f}{\Delta \sqrt{\beta_1}}\cdot\frac{\partial \sqrt{\beta_1}}{\partial g_i} + \frac{\Delta P_f}{\Delta \beta_2}\cdot\frac{\partial \beta_2}{\partial g_i}$$

- Differentiation using chain rule.
- $\frac{\Delta P_f}{\Delta \mu}, \frac{\Delta P_f}{\Delta \sigma}, \frac{\Delta P_f}{\Delta \sqrt{\beta_1}}, \frac{\Delta P_f}{\Delta \beta_2}$ can be calculated by

by the Pearson system and finite difference method.

Influence Index

- Cont'd

$$\mu = \sum_{i=1}^{n} w_i g_i, \quad \frac{\partial \mu}{\partial g_i} = w_i$$

$$\sigma = \sqrt{\sum_{i=1}^{n} w_i (g_i - \mu)^2}, \quad \frac{\partial \sigma}{\partial g_i} = \frac{w_i}{\sigma}\left((g_i - \mu) - \sum_{j=1}^{n} w_j^2 (g_j - \mu) \right)$$

$$\sqrt{\beta_1} = \frac{\sum_{i=1}^{n} w_i (g_i - \mu)^3}{\sigma^3}, \quad \frac{\partial \sqrt{\beta_1}}{\partial g_i} = 3 w_i \cdot \frac{(g_i - \mu)^2 - \sum_{j=1}^{n} w_j^2 (g_j - \mu)^2}{\sigma^3} - 3 \cdot \frac{\frac{\partial \sigma}{\partial g_i} \sum_{j=1}^{n} w_j (g_j - \mu)^3}{\sigma^4}$$

$$\beta_2 = \frac{\sum_{i=1}^{n} w_i (g_i - \mu)^4}{\sigma^4}, \quad \frac{\partial \beta_2}{\partial g_i} = 4 w_i \cdot \frac{(g_i - \mu)^3 - \sum_{j=1}^{n} w_j^2 (g_j - \mu)^3}{\sigma^4} - 4 \cdot \frac{\frac{\partial \sigma}{\partial g_i} \sum_{j=1}^{n} w_j (g_j - \mu)^4}{\sigma^5}$$

Update of response surface

- Experiment is added one by one.
- Mixed term $x_i x_j$ can be added to RS formulation

$$\tilde{g}(\mathbf{x}) = a + \sum_{i=1}^{n} b_i x_i + \sum_{i=1}^{n} c_i x_i^2 + \sum_{k=1}^{nmix} d_k x_{i(k)} x_{j(k)}$$

- Which term should be added?
  - Consider the level combination of the newly added experiments,

$x_1(0):x_2(1):x_3(0):x_4(-1):x_5(1)$

Candidates : $x_2 \cdot x_4$, $x_2 \cdot x_5$, $x_4 \cdot x_5$

Update of response surface

- Cont'd
  - Select a mixed term $x_i \cdot x_j$ that has greatest value of

$$cs_{ij} = |b_i| + |c_i| + |b_j| + |c_j|$$

  - If the selected term is already included in the formulation, then select $x_i \cdot x_j$ with the next biggest cs_{ij}
  - More than one term can be added.

Convergence check

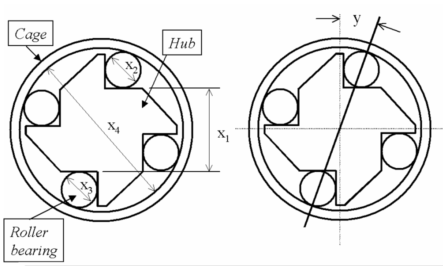
- Convergence criteria

$$\left| \frac{P_f - P_{f0}}{P_{f0}} \right| < \varepsilon$$

  - If the convergence criterion is satisfied for 3 consecutive experiments, then algorithm terminates.

Ex. 1 : Fortini's Clutch

Fortini's Clutch   Greenwood and chase, 1990



System response function

$$y = \arccos\left[\frac{x_1 + \frac{1}{2}(x_2 + x_3)}{x_4 - \frac{1}{2}(x_2 + x_3)}\right]$$

System requirement

$$5^\circ \le y \le 9^\circ$$

Component	$f(\cdot)$		Mean	STD	Parameter for nonnormal distribution
x_1	normal	beta	$55.29mm$	$0.0793mm$	$\gamma_1 = \eta_1 = 5.0$
x_2	normal	normal	$22.86mm$	$0.0043mm$	$(55.0269 \le x_1 \le 55.5531)$
x_3	normal	normal	$22.86mm$	$0.0043mm$	$\hat{\sigma}_4 = 0.1211$
x_4	normal	Rayleigh	$101.60mm$	$0.0793mm$	$(x_4 \ge 101.45)$

Ex. 1 : Fortini's Clutch

First approximation ($y \le 6^\circ$)

Number of experiments at first approximation : 9
Number of terms in RS approximation : 9
RS coefficient at first approximation

0.12506456	a
-0.01320876	b_1
-0.00077123	b_2
-0.00077123	b_3
0.01383499	b_4
-0.00071070	c_1
-0.00000245	c_2
-0.00000245	c_3
-0.00080407	c_4

mean : 0.12193950
std : 0.01159093
skewness : 0.09206418
kurtosis : 2.90359915
pr_failure : 0.06664917

	Proposed Method		FDM (Δ=0.001)		Error
	$\left\lvert\frac{dP_f}{d\tilde{g}_i}\right\rvert$	Exp. No.	$\left\lvert\frac{\Delta P_f}{\Delta\tilde{g}_i}\right\rvert$	Exp. No.	
1	1.476153	67	1.369570	67	7.5%
2	0.551716	77	0.561110	77	5.7%
3	0.551716	71	0.561110	71	5.7%
4	0.483171	59	0.495354	59	6.9%
5	0.483171	65	0.495354	65	6.9%
…	……	……	……	……	……

Ex. 1 : Fortini's Clutch

Second approximation

experiment 67 is executed additionally.
Number of experiments : 10
Term x 1* x 4 is included in exp. coordinate
Term x 1* x 4 is added
Number of terms in RS approximation : 10
RS coefficient at first approximation

0.12506456	a
-0.01280947	b_1
-0.00077123	b_2
-0.00077123	b_3
0.01383499	b_4
-0.00071070	c_1
-0.00000245	c_2
-0.00000245	c_3
-0.00080407	c_4
0.00217198	d_3

mean : 0.12193950
std : 0.01161810
skewness : -0.10661204
kurtosis : 2.84110933
pr_failure : 0.07332022

Exp.	$\left\lvert \frac{dP_f}{d\tilde{g}_i} \right\rvert$
71	0.559847
77	0.559847
59	0.502577
65	0.502577
49	0.484110
...	......

Ex. 1 : Fortini's Clutch

Fourth approximation

experiment 59 is executed additionally.
Number of experiments : 13
Term x 1* x 2 is included in exp. coordinate
No term is added
Number of terms in RS approximation : 12
RS coefficient at first approximation

0.12506493	a
-0.01280981	b_1
-0.00077123	b_2
-0.00077123	b_3
0.01383499	b_4
-0.00071123	c_1
-0.00000298	c_2
-0.00000281	c_3
-0.00080444	c_4
-0.00009912	d_1
-0.00009947	d_2
0.00217111	d_3

mean : 0.12193919
std : 0.01161827
skewness : -0.10761175
kurtosis : 2.84166700
pr_failure : 0.07335008

Ex. 1 : Fortini's Clutch

Results

	HL-RF *	3^n moment	RSM moment	MCS (1,000k)
μ_G	•	0.121930	0.121939	0.121926
σ_G	•	0.011687	0.011618	0.011694
$\sqrt{\beta}_{1G}$	•	-0.057661	-0.107611	-0.051593
β_{2G}	•	2.921503	2.841667	2.880998
$\Pr(y \le 5°)$	Diverge	0.001580	0.001459 (13)	0.001288
$\Pr(y \le 6°)$	0.087656 (8,40)	0.072562	0.073350 (13)	0.073922
$\Pr(y \le 7°)$	0.520349 (5,25)	0.504294	0.504847 (13)	0.503160
$\Pr(y \le 8°)$	0.936942 (3,15)	0.936245	0.935951 (12)	0.936726
$\Pr(y \le 9°)$	0.999448 (3,15)	0.999250	0.999288 (12)	0.999190
No. of fn call	(iteration, #)	81	(#)	1,000,000

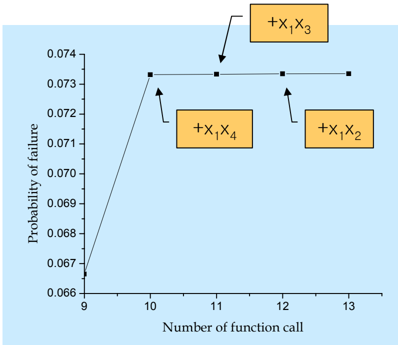
* : 2 iteration schemes and 2 transformation schemes are tried

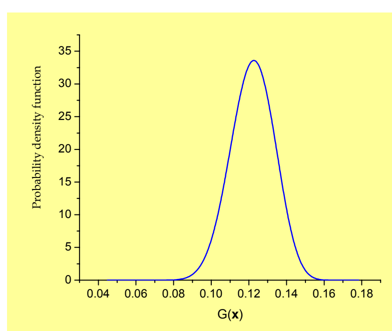
Ex. 1 : Fortini's Clutch

Results

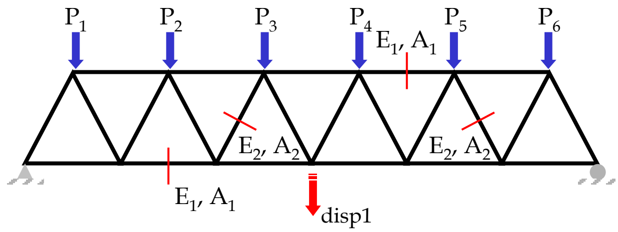
$$\tilde{g}(\mathbf{x}(\boldsymbol{\xi})) = 0.1251 - 0.0128\xi_1 - 0.0008\xi_2 - 0.0008\xi_3 - 0.0138\xi_4 - 0.0008\xi_1^2 - 0.0008\xi_4^2 - 0.0001\xi_1\xi_2 - 0.0001\xi_1\xi_3 + 0.0022\xi_1\xi_4$$

$$\xi_i = \frac{x_i - (l_{i3} + l_{i1})/2}{(l_{i3} - l_{i1})/2}$$





Ex. 2 : Truss Structure



$$g(\mathbf{x}) = 11 - disp1 \le 0$$

Ex. 2 : Truss Structure

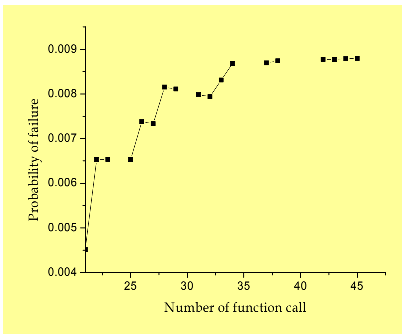
$$\begin{aligned} g(\mathbf{x}(\boldsymbol{\xi})) = 2.8070 &+ 1.2598\xi_1 + 0.2147\xi_2 + 1.2559\xi_3 + 0.2133\xi_4 - 0.1510\xi_5 \\ &- 0.4238\xi_6 - 0.6100\xi_7 - 0.6100\xi_8 - 0.4238\xi_9 - 0.1510\xi_{10} \\ &- 0.1978\xi_1^2 - 0.0362\xi_2^2 - 0.2016\xi_3^2 - 0.0346\xi_4^2 + 0.0023\xi_5^2 \\ &+ 0.0008\xi_6^2 + 0.0036\xi_7^2 + 0.0036\xi_8^2 + 0.0008\xi_9^2 + 0.0023\xi_{10}^2 \\ &- 0.0042\xi_1\xi_2 - 0.3022\xi_1\xi_3 - 0.0110\xi_1\xi_4 + 0.0381\xi_1\xi_5 + 0.0871\xi_1\xi_6 \\ &+ 0.1232\xi_1\xi_7 + 0.1232\xi_1\xi_8 + 0.0871\xi_1\xi_9 + 0.0346\xi_1\xi_{10} + 0.0041\xi_2\xi_3 \\ &+ 0.0110\xi_3\xi_4 + 0.0261\xi_3\xi_5 + 0.0831\xi_3\xi_6 + 0.1172\xi_3\xi_7 + 0.1172\xi_3\xi_8 \\ &+ 0.0832\xi_3\xi_9 + 0.0296\xi_3\xi_{10} \end{aligned}$$

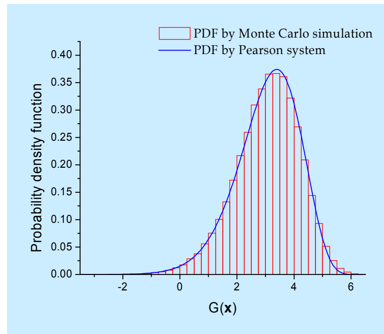
$$\xi_i = \frac{x_i - (l_{i3} + l_{i1})/2}{(l_{i3} - l_{i1})/2}$$

Ex. 2 : Truss Structure

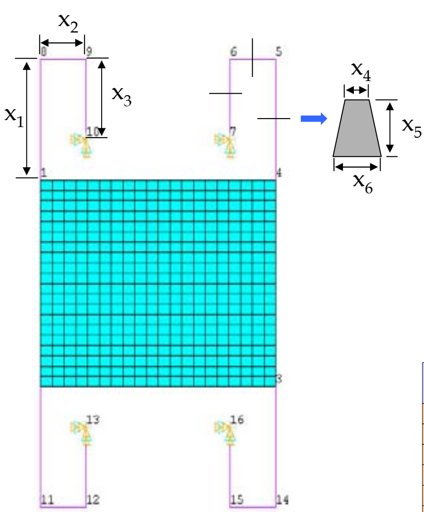
Results

- 23 additional experiments (17 terms are added)
- Pearson type I (Beta distribution)





Ex. 3 : Micro-Gyroscope

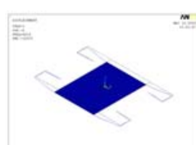


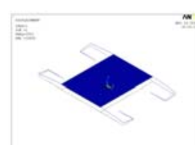
Tanaka, et al, 1995

$g(\mathbf{x}) = f_2 - f_1$ (Hz)

f_1 : First natural frequency

f_2 : Second natural frequency





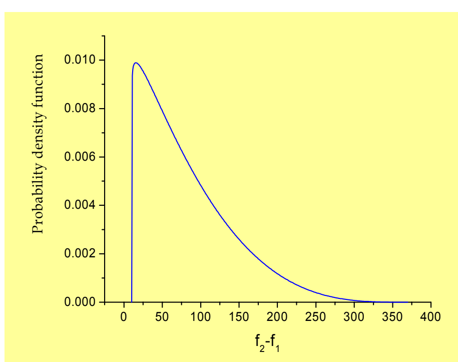
Ex. 3 : Micro-Gyroscope

$$\tilde{g}(\mathbf{x}(\xi)) = 62.7989$$
$$-2.0597\xi_1 + 5.7858\xi_2 - 1.3070\xi_3 + 45.4750\xi_4 - 64.1778\xi_5 + 66.8559\xi_6$$
$$+0.0048\xi_1^2 - 0.0091\xi_2^2 - 0.0009\xi_3^2 + 4.5491\xi_4^2 + 17.7791\xi_5^2 + 32.3739\xi_6^2$$
$$-0.0929\xi_1\xi_4 + 0.0518\xi_2\xi_4 - 0.1056\xi_3\xi_4 - 78.5515\xi_4\xi_5 + 0.0286\xi_4\xi_6 - 36.5115\xi_5\xi_6$$

* : Moment results are for case g(**x**)>100

Ex. 3 : Micro-Gyroscope

- Results
  - 3~7 additional experiments (6 terms are added)
  - Pearson type I (Beta distribution)



Discussion

- RS-Moment method (RSMM)
  - Highly efficient and good accuracy
  - Additional information available from the RS model
  - The accuracy cannot exceed that of 3^n moment method (FFMM)

..

Reliability analysis: Level 2 Methods

FORM

■ AFOSM

Transformation $\mathbf{T}$

$\mathbf{X} = \mathbf{T}\mathbf{u}$

$\mathbf{X}$ → $\mathbf{u}$

- Normal

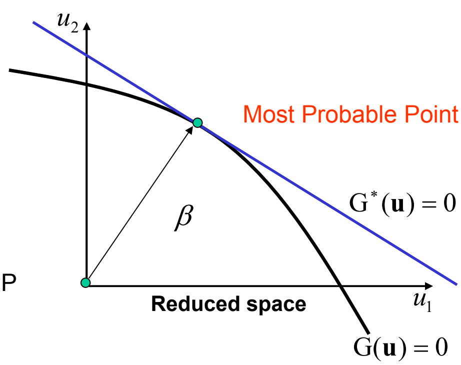
- Independent
- Standard normal

Reliability index (RI) β

Minimum distance from origin to MPP

$$\min \ |\mathbf{u}| = \beta \quad s.t. \quad G(\mathbf{u}) \le 0$$

First-order Taylor expansion of $G(\mathbf{u})$ at MPP



$$P_f = \Pr(G(x) \le 0) \cong \Phi(-\beta)$$

RBDO Formulations with AFOSM

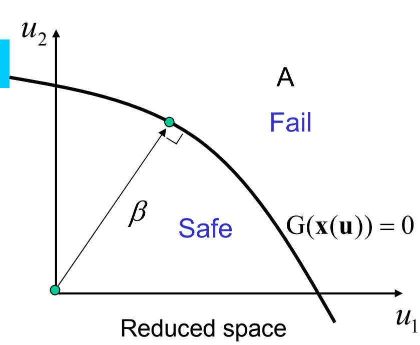
$$\Pr\left[G(\mathbf{b},\mathbf{z},\mathbf{x}) \le 0\right] \le p$$

1. Reliability index approach

$$1 - \Phi(\beta) \le p$$

$$\text{where} \ \beta = \min_{u \in A} (u^T u)^{1/2}$$

$$A = \{u : G(u) \le 0\}$$



Note: 1) A is dependent on $\mathbf{b}$, $\mathbf{z}$ and $\mathbf{x}$

2) For large β, difficult to obtain good results

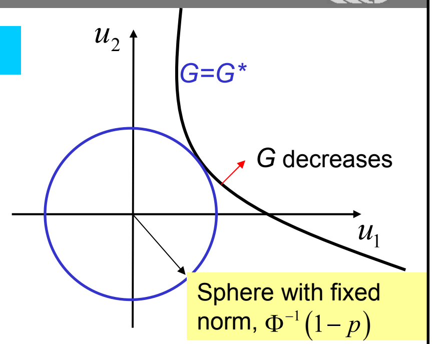
RBDO Formulations with AFOSM

2. Fixed norm approach (FNA)

$$G^* \geq 0$$

$$G^* = \min_{\mathbf{u} \in B} G(\mathbf{x}(\mathbf{u}))$$

$$\text{where } B = \left\{ \mathbf{u} \,\middle|\, \left(\mathbf{u}^{\mathrm{T}}\mathbf{u}\right)^{\frac{1}{2}} \leq \Phi^{-1}(1-p) \right\}$$



Note:

1) B is a sphere with a fixed norm in the reduced random variable space and independent of b and z. Even for large β, this works well

2) This transformed formulation, FNA, first developed by [Lee & Kwak, 1987-88]

3) Much later, [Tu & Choi, 1999] presented the same formulation by the name PMA (Performance Measure Approach). They did not cite our original work though.

Sensitivity of probability

Sensitivity Analysis [Lee & Kwak, 1987]

$$\frac{d\hat{G}_j}{d\mathrm{b}} = ?$$

$$\min \quad W(\mathbf{b})$$

$$H_i(\mathbf{b}, \mathbf{z}, \mathbf{x}) = 0 \quad i = 1, \cdots, s$$

$$\Pr\left[\bigcup_{j=1}^{m} \{G_j(\mathrm{b}, \mathrm{z}, \mathrm{x}) \leq 0\}\right] \leq p_0$$

$$\Pr\left[G_j(\mathrm{b}, \mathrm{z}, \mathrm{x}) \leq 0\right] \leq p_j, \; j = m+1, \ldots, m'$$

$$G_j(\mathrm{b}) \leq 0, \quad j = m'+1, \ldots, m''$$

$$\hat{G}_j(b) = \min_{\mathbf{u},\mathbf{z}} \quad G_j(\mathbf{b}, \mathbf{z}, \mathbf{u}) \geq 0$$

$$\mathbf{H}(\mathbf{b}, \mathbf{z}, \mathbf{u}) = 0$$

$$Q_j(\mathbf{b}, \mathbf{u}) \leq 0$$

$$\delta G_j = \left[\frac{\partial G_j}{\partial \mathbf{b}} + \boldsymbol{\lambda}^T \frac{\partial \mathbf{H}}{\partial \mathbf{b}} + \mu \frac{\partial Q_j}{\partial \mathbf{b}}\right] \delta \mathbf{b}$$

Parametric optimal design (POD) [Kwak & Haug, 1977]

OD procedure using sensitivity

- Sensitivity analysis
  - Sensitivity of probability by POD
  - FDM sensitivity analysis using DOE
  - Original FDM → too expensive to use
- Any first order gradient method can be used.
- Example application area
  - Tolerance analysis and synthesis

Application: Tolerance Design

- Tolerance Analysis (Seo & Kwak 2002)

$$y_i = g_i\left(x_1, x_2, \ldots, x_n\right) \qquad \left(i = 1, 2, \ldots, m,\ y_i = \bar{y}_i + u_i,\ x_j = \bar{x}_j + t_j\right)$$

$$\Delta y_i = \sum_{j=1}^{n} \frac{\partial g_i}{\partial x_j} \Delta x_j + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 g_i}{\partial x_j \partial x_k} \Delta x_j \Delta x_k + \cdots \approx \frac{\partial g_i}{\partial x_1} \Delta x_1 + \frac{\partial g_i}{\partial x_2} \Delta x_2 + \cdots + \frac{\partial g_i}{\partial x_n} \Delta x_n$$

  - Worst case method
  - RSS technique
  - FORM / SORM
  - Monte Carlo simulation
  - **DOE**

Tolerance Synthesis based on RBDO Concept

- Tolerance is a function of probability distribution parameters of dimensions.
  For example, for 3σ quality, $\sigma = t/3$.
- System characteristics, $g(x)$, is a function of the random variables (dimensions), x, thus a function of tolerance, t.
- Functional requirement or quality may be described by inequality, $g(x) \geq 0$. This needs to be satisfied with some certainty.
- A tolerance allocation problem is then a problem of balancing between the cost and quality. This results in the same formulation as an RBDO.

Tolerance Synthesis Problem Formulation

$$\begin{aligned} &\textit{Minimize } \sum_i C_i(t_j) \\ &\textit{subject to } \Pr_{safe} \geq \Pr_{Prescribed} \end{aligned}$$

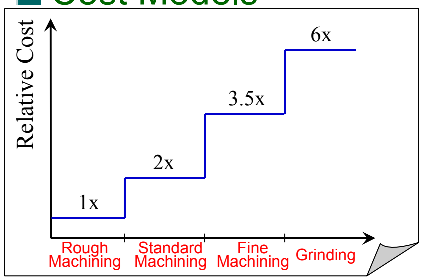
or

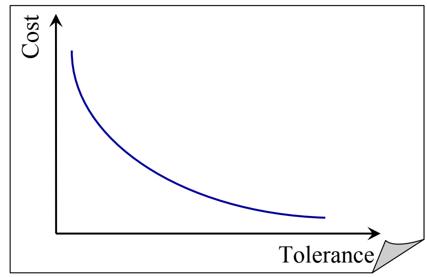
$$\begin{aligned} &\textit{Maximize } \Pr_{safe} \\ &\textit{subject to } \sum_i C_i(t_j) \leq C_{Prescribed} \end{aligned}$$

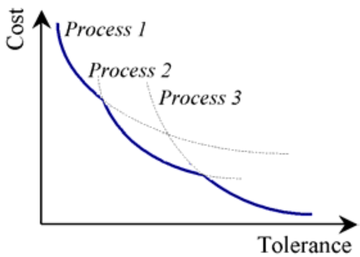
$$\textit{where } \Pr_{safe} = \Pr(g(\mathbf{x}) \geq 0)$$

Tolerance Optimization

Cost Models







Examples of Cost functions

$$A+\frac{B}{t^r},\qquad A+\frac{B}{e^{st}}$$

Design Sensitivity Analysis using DOE

- Probabilistic constraint : $\Pr_{safe}=\Pr\left[g(\mathbf{x})\ge 0\right]\ge Y_{spec}$
- Sensitivity : $d\Pr_{safe}/d\mathbf{x}$

$\Pr_{safe}$	: Function of $\mu_g,\sigma_g,\sqrt{\beta_{1g}},\beta_{2g}$
$\mu_g,\sigma_g,\sqrt{\beta_{1g}},\beta_{2g}$	: Function of **l**(level),**w**(weight)
l, **w**	: Function of $\mu_{x_k},\sigma_{x_k},\sqrt{\beta_{1x_k}},\beta_{2x_k}$
$\mu_{x_k},\sigma_{x_k},\sqrt{\beta_{1x_k}},\beta_{2x_k}$	: Distribution type of variable x_k

⇒

$\partial\Pr_{safe}/\partial\mu_g,\cdots$

$\partial\mu_g/\partial l_{i_k,k},\partial\mu_g/\partial w_{i_k,k},\cdots$

$\partial l_{i_k,k}/\partial\mu_{x_k},\partial w_{i_k,k}/\partial\mu_{x_k},\cdots$

$\partial\mu_{x_k}/\partial x_k,\cdots$

- Derivative of performance function value : use previously obtained DOE data and chain rule

$$\frac{\partial\mu_g}{\partial l_{i_k,k}},\frac{\partial\sigma_g}{\partial l_{i_k,k}},\frac{\partial\sqrt{\beta_{1g}}}{\partial l_{i_k,k}},\frac{\partial\beta_{2g}}{\partial l_{i_k,k}}\Rightarrow\frac{\partial g(\mathbf{l})}{\partial l_{1,k}},\frac{\partial g(\mathbf{l})}{\partial l_{2,k}},\frac{\partial g(\mathbf{l})}{\partial l_{3,k}}$$

Example

Verification with the Fortini Clutch example

- Constraint

$$G(\mathbf{t}) = \Pr_{spec} - \Pr\left[5^\circ \le y(\mathbf{x}) \le 9^\circ\right] \le 0, \qquad where \;\; \mathbf{x} = \boldsymbol{\mu} + \mathbf{t}$$

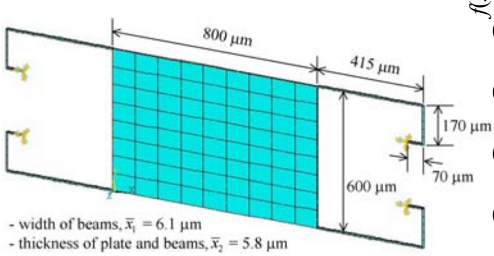
- Comparison of the sensitivity results with FDM

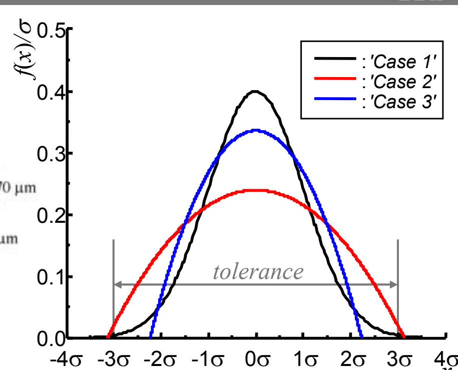
i	Sensitivity, $dG(\mathbf{t})/dt_i$	
	The proposed method	FDM*
1	0.026416	0.026871
2	0.002059	0.002059
3	0.002059	0.002059
4	0.029027	0.029490

* In FDM, $\Delta\mathbf{t} = 1\times10^{-5}\,\mathbf{t}$

Example

Micro-Gyroscope





$$Minimize \;\; \left(\frac{2\times10^{-2}}{t_1^{1.3}} + 200\right) + \left(\frac{3\times10^{-2}}{t_2^{1.1}} + 200\right)$$

$$subject \;\; to \qquad \Pr\left[0 \le f_2 - f_1 \le 200\right] \ge 0.95$$

- **Case 1 :**
  normal distribution : s.t.d. = σ
- **Case 2 :**
  beta distribution : $DPMO_2$ = $DPMO_1$
- **Case 3 :**
  beta distribution : s.t.d. = σ

Examples

- Sensitivity results

Tolerance type	The proposed method		FDM		
	$dG(\mathbf{t})/dt_1$	$dG(\mathbf{t})/dt_2$	Δt	$dG(\mathbf{t})/dt_1$	$dG(\mathbf{t})/dt_2$
Case 1	86.207442	389.29906	10^{-3}	86.313433	389.49592
			10^{-4}	86.225993	388.94631
			10^{-5}	84.941866	383.90791
Case 2	132.30349	721.93610	10^{-3}	132.48322	721.94789
			10^{-4}	132.41351	721.79235
			10^{-5}	132.80005	717.79549
Case 3	118.13797	467.51404	10^{-3}	1.1823083	4.6788787
			10^{-4}	1.1830852	4.6745320
			10^{-5}	1.1883938	4.6401483

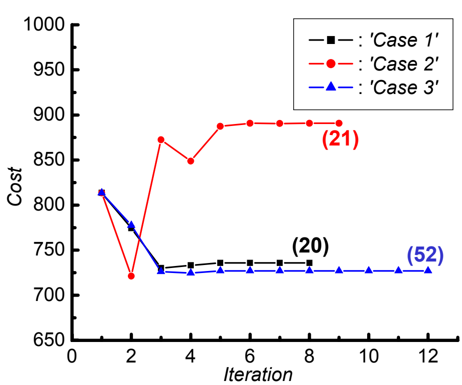
- Optimization results

Variable	The proposed method			FDM		
	Case 1	*Case 2*	*Case 3*	*Case 1*	*Case 2*	*Case 3*
t_1	0.944μm	0.698μm	0.969μm	0.946μm	0.698μm	0.978μm
t_2	0.399μm	0.286μm	0.406μm	0.398μm	0.286μm	0.401μm
$\sum C_i$	735.987	890.665	726.984	735.805	890.666	727.047

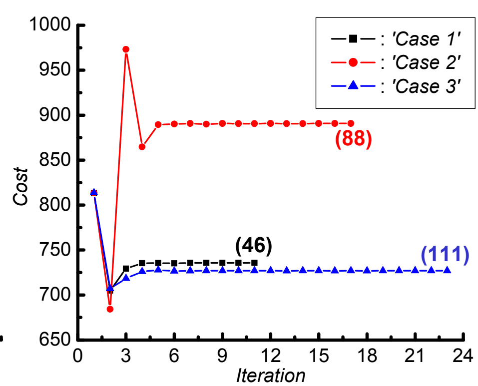
Example

- Optimization history

• by the proposed method



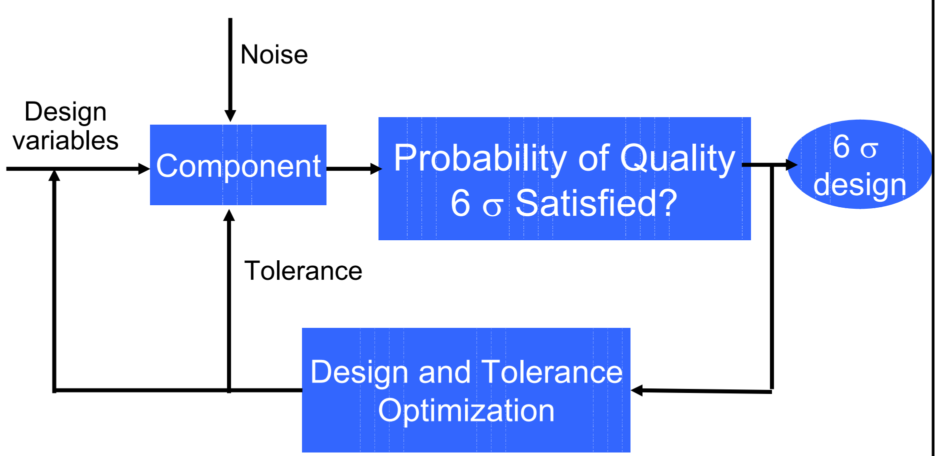
• by FDM



Summary

- Moment based DOE reliability analysis using 3-point probability concentration and Pearson system is developed for treating general distributions.
- Formal optimal design procedure for RBDO is developed with very good accuracy and applied to tolerance analysis and synthesis using previously obtained DOE data.
- Expanding response surface moment method (RSMM) is developed to drastically reduce the amount of computation even for a large number of random variables.
- Design for 6-sigma is possible with the methods developed.

Design Optimization for 6 Sigma



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