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AXIOMATIC APPROACH IN THE ANALYSIS OF DATA FOR DESCRIBING COMPLEX SHAPES

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ABSTRACT

In this paper the axioms, of Axiomatic Design, are extended to the non-probabilistic and repetitive events. The idea of information, in the theories of Fisher and Wiener-Shannon, is a measure only on probabilistic and repetitive events. The idea of information is larger than the probability. It is possible the formulation of an Extended Theory of Information for probabilistic and non-probabilistic events. As example is studied an application in which the number of DPs is less then the number of FR_s, and the coupled design cannot be satisfied.

Keywords: design, axiomes, information, non-probabilistic information

1 INTRODUCTION

The design process gives the structure necessary for the transformation of the qualitative needs, often stated in non engineering terms, to the real products.

This transformation is achieved through the application of scientific knowledge to the problems. Using previous design databases the design process generate several alternatives to be evaluated frequently.

Usually the design process is subdivided into a series of phases with specific, evaluations to make between these phases. Each evaluation determines whether the phase needs to be repeated, or if the designer needs to go back on one or more phase.

Nam P. Suh (1990) proposes an axiomatic method for highly complex designs. The design process optimizes elements using a set $\{FR_i\}$ of functional requirements and a set $\{DP_j\}$ of physical parameter.

He proposed two axioms that could help to have a good design. The relation of $\{FR_i\}$ with the $\{DP_j\}$ is mathematically expressed as

$$FR_i = f\left(DP_j\right)$$

The design process is reduced to a series of mappings from the design's functional requirements into the design's parameter space. The mapping process between the domains is repeated several times, with the results that previous design parameters determine the next set of functional requirements.

The domains are defined by the vectors:

$$\{FR\} = vector\ \ of\ \ functional\ domain$$
$$\{DP\} = vector\ of\ \ physical\ domain$$
$$\{PV\} = vector\ of\ \ process\ domain$$

The relation between these two domain in matrix notation is written as

$$\{FR\} = [A]\{DP\} \quad (1)$$
$$\{DP\} = [B]\{PV\} \quad (2)$$

were $[A]$ and $[B]$ are the design matrices.

In the problems in which the $\{FR_i\}$ depend from non linear functions, the equation (1) can be written in differential form

$$\{dFR\} = [A]\{dDP\}$$

The elements of design matrix can be written as $A_{ij} = \partial FR_i / \partial DP_j$ and the design matrix is

$$[A]=\begin{bmatrix} \partial FR_1/\partial DP_1 & .. & .. & \partial FR_1/\partial DP_j \\ .. & .. & .. & .. \\ \partial FR_i/\partial DP_1 & .. & .. & \partial FR_i/\partial DP_j \end{bmatrix}$$

A small change in any parameter may cause a deviation in the functional requirement

$$\Delta FR_i = \frac{\partial FR_i}{\partial DP_j}\Delta DP_j$$

In linear design $A_{ij} = \partial FR_i / \partial DP_j$ are constants. We have

$$FR_i = \sum_{i=1}^{n} A_{ij} DP_j$$

In (1), the diagonal matrix is a special case. The design matrix $[A]$, in general, is a rectangular array of values.

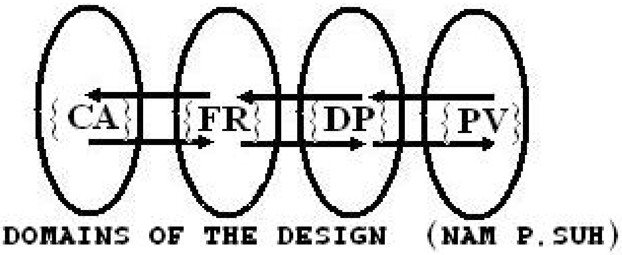


Figure1

In the design process Nam P. Suh has indicated two axioms, on functional requirement, in order to examine the actions of planning. The axioms are:

Axiom 1: The independence Axiom. *Maintain the independence of functional requirement.*

Axiom 2: The information Axiom. *Minimize the information contents of the design*.

The first axiom states that the independence of functional set $\{FR_i\}$ must be always maintained. The information axiom states that the best design has the minimum information and the minimum of functional requirements. For comparing two design, one can compare the information content of the two design which can satisfy the functionally parameters. The information content can be described by means similar to the Wiener-Shannon's theory.

The elements of a design matrix $[A]$ can be constants or functions, with the consequence that the design may be non-linear. Mathematical techniques can transform a matrix, but the physical significance of the elements A_{ij} can be lost. An ideal design matrix is a square diagonal matrix with each *FR* related one to one to a single *DP*. The uncoupled tolerance for a DP_i is $\Delta DP_i = \Delta FR_i / A_{ii}$. The propagation of tolerance for a decoupled design with a lower triangular matrix $n \times n$, is expressed as

$$\Delta DP_i^* = \frac{\Delta FR_i - \sum_{j=1}^{n} \left| A_{ij} \Delta DP_j \right|}{A_{ii}}$$

From equation (2) it is evident that is $\Delta DP_i \geq \Delta DP_i^*$. The consequence is that decoupled design has less tolerance than an uncoupled design, and the increase of the order of design matrix makes the last DP_i's tolerance smaller. If the number of DP_s is greater than the number of FR_s, then the design is redundant.

When the number of DP_s is less then the number of FR_s, then the coupled design cannot be satisfied. Suppose that there is a set of three $\{FR_1, FR_2, FR_3\}$ and a set of two $\{DP_1, DP_2\}$, then the equation in matrix notation is

$$\begin{Bmatrix} FR_1 \\ FR_2 \\ FR_3 \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} \begin{Bmatrix} DP_1 \\ DP_2 \end{Bmatrix} \quad (3)$$

In this equation FR_3 cannot be always satisfied. The equation (3) can be written

$$\begin{aligned} FR_1 &= A_{11}DP_1 + A_{12}DP_2 \\ FR_1 &= A_{21}DP_1 + A_{22}DP_2 \\ FR_1 &= A_{31}DP_1 + A_{32}DP_2 \end{aligned} \quad (4)$$

It is not possible to have a solution of system (4) without to make changes to functional requirements.

2 DESIGN WITH THE NUMBER OF DPS LOWER THAN THE NUMBER OF FR2

In ideal design each functional requirement must be linked to one design parameter, and vice versa each design parameter can satisfy one (or more) functional parameter. From the system of equations (4) it turns out obvious that with the number of $DP_s < FR_s$ it is possible to have only approximate solutions. In this situation the number of DP_S is insufficient to achieve all the FR_s in exact mode. A good design has the minimum information content. Analyzing the information of a design it is possible to understand the physical influence of constraints. The information axiom states that the best design has then minimum information and the minimum of functional requirements. It is useful for to choose between two or more designs. In absence of solution we cannot compare anything: w*e need at least a solution*.

Using mathematical transformations it is possible to obtain an approximate solution. If a set of tolerances is imposed to the domain of FR_s , then the tolerances are propagated from domain to domain and the set of DP_s will be modified.

If we use the idea of information in metric space, using the Laplace's principle of insufficient reason, in according with Maximum Entropy Principle of Jaynes (*MaxEnt* Principle), we can select as solution the distribution that maximize the Shannon entropy measure and simultaneously is consistent with the values of constraints. With *MaxEnt* Principle it is possible to have a solution when the number of DP_s is less than the number of FR_s .

Now it is important to understand the physical significance of mathematical transformations deriving from the use of the Jaynes's principle

The idea of information, in the theories of Fisher and Wiener-Shannon, is a measure only of probabilistic and repetitive events. The idea of information is larger than the probability and the axioms of Wiener–Shannon can be extended to the non-probabilistic and repetitive events. Let Ω to be the field of all events ω (*Fig. 2*), probabilistic or non-probabilistic, and $\Im$ a class of parts of Ω , $\Im \subset \wp_{arts}(\Omega)$. With $A \subset \Im$ we can assume the next two axioms:

AXIOM I: The value of information *J(A)* is always a number non-negative*:*

$$J(A) \; : \; \Im \rightarrow \mathbf{R}^+ \qquad (5)$$

AXIOM II: The value of information *J(A)* is monotonous in regard to inclusion:

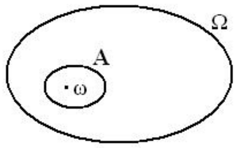
$$\forall A, B \in \Im \;,\; B \subset A \;,\; J(B) \geq J(A) \qquad (6)$$

Now it is possible the construction of new algorithms in terms of information, founded only on the first and second axioms [3]. For independent events it is opportune to assume a third axiom:

AXIOM III: If the events $A, B \in \Im$ are independent for all the values of information, we have:

$$\forall A, B \in \Im \quad J(B \cap A) = J(B) + J(A) \quad (7)$$

The third axiom shows that when we are in presence of independent events it is possible to add up information.



Fifure 2

If Ω is a certain event and ϕ the impossible event than, for an universal validity of $J(A)$ and $J(\phi)$, for all $\Omega, \Im$ and J must be:

$$J(\Omega) = 0 \;,\; J(\phi) = +\infty \qquad (8)$$

The expression $J(\Omega) = 0$ means that Ω is a certain event without need of information. The expression $J(\phi) = +\infty$ means that if ϕ is an impossible event with the need of infinite information. In a metric space Ω, if ω is an event in $\Im \subset \wp_{arts}(\Omega)$, its measure will be always incorrect. The knowledge of ω is not given by its coordinates in Ω , but it is possible only to assert that ω is limited in a subset $A_i \in \Im$. If $d(A_i)$ is the diameter of set A_i , than, more is the precision of measures, less is the measure of diameter of event A_i . If we assume that $\{P_{x,y}\}$ is a set of ideal data in a continuous closed bounded subset $\Omega \in [D]$, given any $\varepsilon > 0$, there is a set $\{M_{x,y}\}$of values of measures with sufficiently high precision such that

$$\left|P_{x,y} - M_{x,y}\right| < \varepsilon \quad \text{for } (x,y) \in \Omega \qquad (9)$$

But the probability *p* of an exact measure is in inverse proportion to the precision, so the ideal measure of point's coordinates has null probability to be obtained: it is an impossible event. The impossible event ϕ and the certain event Ω are always independent from J and *A*: they are universal values. All three axioms have correspondent axioms in Wiener-Shannon theory.

With the axioms $J(\Omega) = 0$ and $J(\phi) = +\infty$ it is possible to construct models for information very useful in applications. In metric space Ω, for every event $A \in \Im$ we can have a measure of information using the mathematical expression :

$$J(A)=\frac{1}{d(A)} \qquad (10)$$

This definition of information has a natural application in metric space [3]. In a metric space Ω, if ω is an event in $\Im \subset \wp_{arts}(\Omega)$, better must be the result of its measure than smaller is the diameter of A_i, and larger will be *J(A)*[9]. If we assume that all the measures are made with equal care, and for any value of ω the data have a normal distribution, the probability that the error $d(A_i)$ will fall in a small interval δ_y is given for ω_i

$$P(\omega_i)=\frac{1}{\sigma\sqrt{2\pi}}\left[\exp-\left(d(A_i)\right)^2/2\sigma^2\right]\delta_y \qquad (11)$$

Similar expression can be written for all ω_i in Ω. The standard deviation σ is a measure of precision of the measurements and it is a constant for all the data. As the separate measurements are independent for all events, the probability for all is the product

$$P=\prod_{k=0}^{N}P(\omega_i)=\left\{\left(\frac{1}{\sigma\sqrt{2\pi}}\right)\left[\exp-\frac{1}{2\sigma^2}\sum_{i=0}^{N}\left(d(A_i)\right)^2\right]\delta_y\right\}^{N+1} \qquad (12)$$

Maximum of P is the sign of the goodness of the measures of the diameters of subsets A_i. This will occur when

$$\sum_{i=0}^{N}\left(d(A_i)\right)^2 = \text{minimum} \qquad (13)$$

In general, the criterion that the sum of diameters $\sum_{i=0}^{N}\left|d(A_i)\right|$ of sets A_i must be as small as possible would have given the better result. We have that the same information can be valued by the probability and by the non-probabilistic measures of diameters. So we can have the measure of information from non-probabilistic data. Now, for to ascribe some information to the realized event A_i, we can assume as measure of information a non-probabilistic function

$$J(A)\overset{def}{=}\Psi\left[\sum\left|d(A_i)\right|\right]$$
$$J(A_i)\propto\frac{1}{\left|d(A_i)\right|} \qquad (14)$$

Any vector $\bar{x}=(x_1,x_2,\cdots,x_n)$ representing proportions of some whole is subject to the unit sum constraints $\sum_i x_i = 1$. One of most usual dissimilarities and distance $d(x_i,x_j)$ to measure the difference between two compositions are Minkowski's distances. In general, it is possible to have the measures of information for probabilistic and non-probabilistic events using empirical or non-empirical functions non-attached to the probability and to the repetitiveness.

It is possible to define a principle for metric space on the ground of the Theory of Information.

On the analogy of *MaxEnt principle*, the name can be *Max Metric Information Principle* (*MaxMetricInf*). Instead of probability, it is possible to utilize a finite number of appropriate proportion subject to a set of constraints that add up to one. In observance of the Axioms, let $d_1,d_2,\ldots,d_n$ be *n* non-negative real numbers, let

$$\sum_{i=1}^{n}d_i \neq 0\ ;\ \rho_i=\frac{d_i}{d_1+d_2+\ldots+d_n}$$
$$\sum_{i=1}^{n}\rho_i = 1\ ; \qquad \forall i\ ; \qquad \rho_i \geq 0 \qquad (15)$$

We can use as the measure of information the relation

$$J(\rho)=J(\rho_1,\ \ \rho_2,\ldots,\ \ \rho_n)=-\sum_{i=1}^{n}\rho_i\ln\rho_i \qquad (16)$$

So that: $J(\rho)$ is maximum when $\rho_1=\rho_2=\ldots=\rho_n$

$J(\rho)$ is minimum when: $\forall i$ only one number is ≠ zero

In metric space, using Euclid's distances, the information maybe

$$\rho^*_{ij}=\left(\frac{\left|d\ (x_i,x_j)\right|}{\sum_{ij}\left|d\ (x_i,x_j)\right|}\right) \qquad (17)$$
$$J=-\sum_{ij}\rho^*_{ij}\log\rho^*_{ij}$$

The value of information $J(\rho)$ is a measure of equality of numbers among themselves. Appling the same formalism of MaxEnt Principle it is easy to define the MaxInf Principle on the basis of so-called *Laplace's Principle* of insufficient reason.

Max Metric Information **:** *<u>Out of all knowledge, choose the solution nearest to the uniform distribution of information.</u>*

In the situations in which we do not have reasons to prefer a solution, it is better choose the solution with uniform distribution, or the closet to the uniform distribution of information.

3 APPLICATION TO AXIOMATIC DESIGN

One application of the *MaxMetricInf* principle is in problems of approximation as criteria to find polynomials for to represent a given set $E = \{(x_i, y_i), ..\}$ of empirical data. Ideally, this process should take in account the reliability of the data, so the more reliable data will have grater weight on approximating function. In absence of knowledge, on basis of *MaxInf* principle, we must use a polynomials, which in representing data, the deviation from them choose the solution closet the uniform distribution of information.

In axiomatic method the design is obtained from a set $\{FR_i\}$ of functional requirements and a set $\{DP_j\}$ of physical parameter. The set FR_s is function of the set DP_s : $FR_i = f(DP_j)$

The design process is made with a series operations, with which the design's functional requirements are mapped into the design's parameter space. The mapping process has as result that the previous design parameters determine the next set of functional requirements.

Let be $FR_i = f(DP_j)$ the approximating function from which we obtain, from the data, the n deviation $\Delta FR_i = FR_i - f(DP_i)$. The estimator vector is

$$\overline{d} = (\Delta FR_1, \Delta FR_2, .., \Delta FR_i)^T \quad (18)$$

From (15) we have

$$\rho_i = \frac{|FR_i - f(DP_i)|}{\sum_i |(FR_i - f(DP_i))|}$$

From (10) as function for to measure the information we can use the function

$$J = \sum_i \frac{1}{|FR_i - f(DP_i)|} \quad (19)$$

From *MaxInf* we have the max value for J when

$$J_1 = J_2 = ... = J_n \quad \frac{1}{|FR_1 - f(DP_1)|} = ... = \frac{1}{|FR_i - f(DP_i)|} \quad \forall i \quad (20)$$

The max of information is obtained when the approximating function $FR_i = f(DP_j)$ <u>has the same error</u> .

$$\|FR_1 - f(DP_1)\| = ... = |FR_n - f(DP_n)| = \Delta FR_T \quad (21)$$

The estimator vector has the distributions

$$\overline{d} = \Big(\underbrace{\Delta FR_T,\ \Delta FR_T,\ ..\ ,\ \Delta FR_T}_{n}\Big)^T \quad (22)$$

If the function is $FR_i = f(DP_j)$

$$FR_i = \sum_{ij} \frac{\partial FR_{ij}}{\partial DP_{ij}} DP_{ij} \quad (23)$$

The deviations of $FR_i = f(DP_j)$ evaluated at certain abscissa and the given ordinate corresponding to the same abscissa:

$$FR_i - \sum_{ij} \frac{\partial FR_{ij}}{\partial DP_{ij}} DP_{ij} = |\Delta FR_T| \quad (24)$$

From the solution of the linear system $A\overline{x} = \overline{b}$

$$\begin{Bmatrix} FR_1 \\ FR_2 \\ \vdots \\ FR_n \end{Bmatrix} = [A] \begin{Bmatrix} DP_1 \\ DP_2 \\ \vdots \\ \Delta FR_T \end{Bmatrix}$$

we have the value of ΔFR_T from which can be valuated the approximation with the max information.

4 EXAMPLE

The example is a case study of the path of the point P which go from A to B (or to C), and from B to C using a linear path, with the condition that the three points, A,B and C, are not on one straight.

The FR_s are

$$\begin{cases} \text{FR1: Possibility of one linear path from A to B} \\ \text{FR2: Possibility of one linear path from A to C} \\ \text{FR3: Possibility of one linear path from} \\ \quad \text{A to B and from B to C.} \end{cases} \quad (25)$$

The FR_3 cannot be mapped in physical domain: it is impossible to go, using a liner path, from A to C, including B. Design parameter satisfying the functional requirements can be only two: FR_1 and FR_2.

The DP_s are

$$\begin{cases} \text{DP1: Design of path from A to B: vector AB} \\ \text{DP1: Design of path from A to C: vector AC} \end{cases} \quad (26)$$

When the number of DP_s is less then the number of FR_s, then the coupled design cannot be satisfied. There are three $\{FR_1, FR_2, FR_3\}$ and two $\{DP_1, DP_2\}$. The equation in matrix notation is

$$\{FR_1, FR_2, FR_3\}^T = [A]\{DP_1, DP_2\}^T \quad (27)$$

In this equation FR_3 cannot be always satisfied. In linear dependence of FR_s from DP_s the $A_{ij} = \partial FR_i / \partial DP_j$ are constants. We have

$$FR_i = \sum_{i=1}^{n} A_{ij} DP_j$$

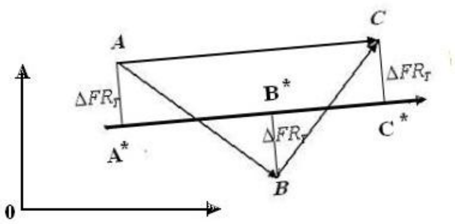


Figure 3

It is possible a solution with the introduction of a tolerance on the presence of points on the path. If it is introduced an *unknown tolerance* ΔDP_T in the design domain then a tolerance ΔFR_T is propagated to the functional domain and on FR_s. If, for the Max Metric Information Principle, is imposed that ΔFR_T is equal for all the elements of $\{FR_s\}$ then the relationship between the two vectors $\{FR_s\}$ and $\{DP_s\}$ can be written

$$\begin{aligned} FR_1 - (A_{11}DP_1 + A_{12}DP_2) &= \Delta FR_T \\ FR_1 - (A_{21}DP_1 + A_{22}DP_2) &= -\Delta FR_T \\ FR_1 - (A_{31}DP_1 + A_{32}DP_2) &= \Delta FR_T \end{aligned} \quad (28)$$

The equation (5) may be written in matrix form

$$\begin{Bmatrix} FR_1 \\ FR_2 \\ FR_3 \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 1 \\ A_{21} & A_{22} & -1 \\ A_{31} & A_{32} & 1 \end{bmatrix} \begin{Bmatrix} DP_1 \\ DP_2 \\ \Delta FR_T \end{Bmatrix} \quad (29)$$

The equation (29) can be easily solved for obtaining the vector of solution *under tolerance* $\{DP_1, DP_2, \Delta FR_T\}^T$. The solution is in figure 3. The value of Design Parameter is the tolerance ΔFR_T.

$$\begin{cases} \text{DP1: Design of path from A to B: vector AB} \\ \text{DP1: Design of path from A to C: vector AC} \\ DP_3 : \text{Design of path from } A^* \text{ to } B^* \text{ to } C^* \\ \qquad \text{with a tolerance } = \Delta FR_T \end{cases}$$

5 CONCLUSION

Using the idea of information, in a larger way than the idea of probability, it is possible the formulation of an Extend Theory of Information for probabilistic and non-probabilistic events. To the question on what happens when the number of DP_s is less than number FR_s it is possible to answer that exist a solution in which function requirement can be satisfied in approximating way.

When number of $DP_s < FR_s$, then the DP_s are insufficient to achieve all the FR_s. If is imposed to the domain of FR_s a set of tolerance, it is possible to carry out a mathematical transformations from which it is possible to obtain all lacking values of DP_s.. The physical significance of mathematical transformation is analyzed with the *MaxEnt* Principle of Jaynes.

The solution, consistent with the values of constraints, is obtained selecting the solution that maximize the Wiener-Shannon information.

In conclusion it is possible to assert that:

When number of $DP_s < FR_s$, using MaxEnt Principle, it is possible to obtain an approximate solution compatible with boundary conditions.

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