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UNDERSTANDING THE VALUE OF ELIMINATING AN OFF-DIAGONAL TERM IN A DESIGN MATRIX

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ABSTRACT

In Axiomatic Design theory, it is strongly desired to eliminate off-diagonal elements from a design matrix as the two design axioms imply an ideal design would have an uncoupled design matrix. For example, in a coupled design, eliminating the off-diagonal term(s) that cause the coupling is mandated by the independence axiom. For a decoupled design, eliminating an off-diagonal term is desirable as it tends to increase robustness and reduces complexity of the design. While eliminating any off-diagonal element does benefit the design in a certain way to a certain degree, each off-diagonal element presents different value when eliminated. For example, a coupled design matrix with a unique structure may be decoupled by eliminating one critical off-diagonal term while a trivial solution may require eliminating more than one off-diagonal terms. Even in a decoupled design, each of the off-diagonal terms when eliminated yields a different FR-DP interaction structure that leads to a different implication in the context of design iteration. This in turn is related to the reduction of imaginary complexity for a design. The fact that eliminating an off-diagonal element among more than one off-diagonal terms is not equally effective in improving a design raises the need for developing an optimal strategy for eliminating off-diagonal element. The first step in developing such strategy is to quantitatively understand the value of (or the lack there of) an off-diagonal element. This paper examines the effect of removing an off-diagonal term to build a basis for an optimum decoupling strategy and presents a preliminary result for the methods to construct an optimal strategy.

Keywords: Axiomatic Design, Design Matrix, Decoupling, Robustness, Design Iteration, Imaginary complexity

1 INTRODUCTION

When conceiving DPs to satisfy a set of FRs, the central objective is to find a set of DPs that satisfy the Independence Axiom and Information Axiom. The Independence Axiom demands that a design maintain the independence of functional requirements. The implication of the Independence Axiom is that a design matrix must follow certain structures to avoid coupling in the design. In other words, off-diagonal elements (a.k.a. coupling terms) of a design matrix should be arranged to have either uncoupled or decoupled structure. In a 3x3 design matrix, for example, there exist a total of six design matrix structures (after rearrangement) that always satisfy the Independence Axiom. All of the uncoupled or decoupled 3x3 design matrices can be rearranged to have one of the following structures:

$$\begin{bmatrix} X & O & O \\ O & X & O \\ O & O & X \end{bmatrix} \quad \begin{bmatrix} X & O & O \\ X & X & O \\ O & O & X \end{bmatrix} \quad \begin{bmatrix} X & O & O \\ X & X & O \\ X & O & X \end{bmatrix}$$

$$\begin{bmatrix} X & O & O \\ O & X & O \\ X & X & X \end{bmatrix} \quad \begin{bmatrix} X & O & O \\ X & X & O \\ O & X & X \end{bmatrix} \quad \begin{bmatrix} X & O & O \\ X & X & O \\ X & X & X \end{bmatrix}$$

Figure 1. Uncoupled or decoupled structures for 3x3 design matrix

Additional off-diagonal terms above the diagonal elements either change its structure to one of the six acceptable structures or make it a coupled design matrix. When a design matrix is coupled, the off-diagonal terms that cause coupling must be eliminated.

While the need to eliminate an off-diagonal element is evident by the Independence Axiom in case of a coupled design, justifying an effort to eliminate an off-diagonal term in a decoupled design needs more careful argument. By definition, a decoupled design satisfy the Independence Axiom only if the structure of the design matrix is known and the sequence dictated by the matrix is properly followed. Thus, eliminating an off-diagonal term does not contribute to improving functional independence of the design. Rather, the value of eliminating an off-diagonal term is in increasing conformity to the Information Axiom and reducing the imaginary complexity of the design.

Existence of an off-diagonal term is considered to compromise a design's robustness, resulting in a lower probability of success [1]. Two main logics behind this argument are 1) reduction in the size of allowable tolerance, and 2) additive effect of an off-diagonal term on FRs' variation.

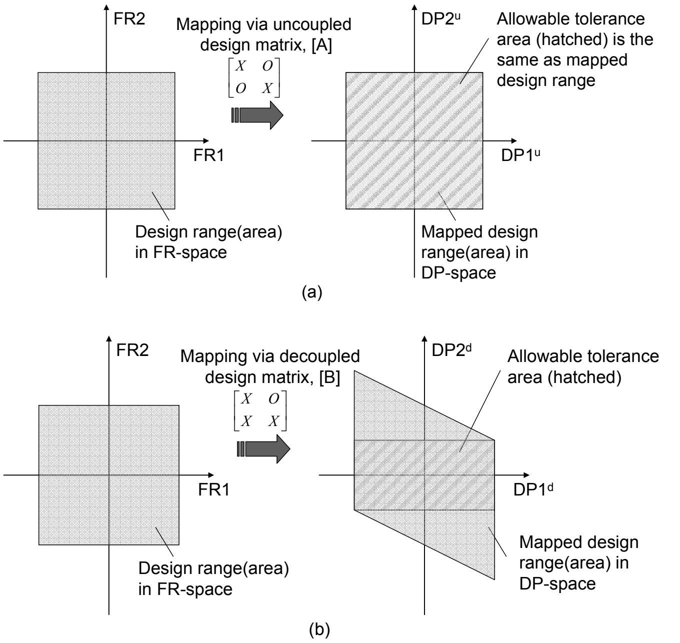


Figure 2. Allowable tolerance (hatched area) for an uncoupled design is larger than that of an equivalent decoupled design. (a) FR design range is mapped onto DP-space as a rectangle, thereby allowing the largest allowable tolerance; (b) a decoupled design matrix maps a rectangular FR design range to be a parallelepiped. Allowable tolerance, shown as a hatched rectangle in (b), is smaller than the area of the mapped-design range.

Theorem 22 and 23 by Suh [1] present the reasoning along the allowable tolerance concept. This concept is best illustrated by a graphical means, shown in Figure 2 [2]. In a 2-FR design problem, design range for the FRs is represented by a rectangle on FR-plane. The design range is always a rectangle since FRs are orthogonal to each other by definition. Assuming a linear design matrix, the rectangular design range can be mapped onto DP-plane. A DP pair ($DP1^i$, $DP2^i$) inside this mapped-design range will yield an FR pair, ($FR1^i$, $FR2^i$), that falls within the design range. In that sense, it can be referred to as a design range for DP. Shape and size of this mapped-design range, or DP design range is a function of the design matrix. If the design matrix is uncoupled, the mapped- design range also bears a rectangular shape with its area scaled by the values of the diagonal terms of the design matrix. If the matrix is decoupled, the mapped-range is a parallelepiped with one axis parallel to either DP1 or DP2 axis. Figure 2(b) shows one of such cases. In a decoupled case, a tolerance for DP1 – where the tolerance is defined as a range for DP within which any value for DP will deliver FR in its design range – is conditional upon a specific value of DP2: the same value of DP1 is within or outside of the mapped-design range depending on DP2 value. Thus, a tolerance for DP1 cannot be assigned independent of DP2. Allowable tolerance is defined in a DP-space to specify the regime where each DP's tolerance can be assigned independent of each other. Finding an unconditional tolerance regime is analogous to finding a rectangle that fits within the mapped-design range, shown as a hatched area in Figure 2. It can be easily shown that for the mapped-design ranges with same area – which in turn requires the determinant of design matrix [A] and [B] be the same –, a rectangular mapped-design range gives the largest allowable tolerance. Since large tolerance for a given design range implies the design is highly robust, we can conclude that an uncoupled design matrix (or the lack of off-diagonal term in general) improves the system's robustness and thereby better conforming to the Information Axiom.

The other part of the benefit of eliminating an off-diagonal term is found in reducing the imaginary complexity of the design. Imaginary complexity in the Axiomatic Design theory is defined as an uncertainty in achieving FRs due to the lack of knowledge on the structure of the design matrix. In other words, it is the complexity caused by the ignorance of the interaction structure between the FRs and DPs. If these interactions are not properly managed, additional design iteration will occur, wasting valuable resource, and the design problem is perceived to be complex. Interactions between FRs and DPs are represented by a design matrix, DM. Diagonal terms are always non-zero by definition of FR-DP mapping. An off-diagonal term can be either zero or non-zero. A non-zero off-diagonal term, DM(i,j), indicates that there is a secondary effect from DPj to FRi while DPj is intended to primarily target FRj. In the presence of such interaction, design process must be conducted such that the interaction is properly taken into account. For example, in a 2FR-2DP design problem, if there is a non-zero off-diagonal element as shown in Figure 3, designer(s) must proceed with the FR1-DP1 design task first and then FR2-DP2 design after observing the effect of DP1 onto FR2. If designer(s) is not aware of the interaction and happens to start from FR2-DP2 design step, then there will be additional design step to complete. The additional design steps that could be avoided if the interaction structure were known a priori is perceived as the problem's complexity, which is defined as the imaginary complexity. The imaginary complexity concept is particularly applicable to a decoupled design because the decoupled design matrix dictates a (set of) unique sequence to be followed during the design process.

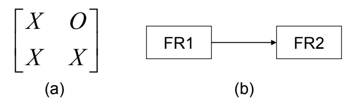


Figure 3. Design matrix represents design interaction structure and dictates a design process. (a) Decoupled design matrix (b) designing FR1-DP1 must precede FR2-DP2

While the benefit of eliminating an off-diagonal term is evident by one or a combination of the reasons explained above, it should be noted that each off-diagonal element presents different value when eliminated. Eliminating an off-diagonal element when there are more than one off-diagonal terms is not equally effective in improving a design. Removing an off-diagonal term takes effort and resource, and we certainly want those resources to be best spent. This raises the need for developing an optimal strategy for eliminating off-diagonal elements. The first step in developing such strategy is to quantitatively understand the value of (or the lack there of) an off-diagonal element. This paper examines the effect of removing an off-diagonal term to build a basis for an optimum decoupling strategy, and presents a preliminary study on the methods to design such strategy.

2 VALUE OF OFF-DIAGONAL ELEMENT IN A DESIGN MATRIX

Within the Axiomatic Design framework, the effort to eliminate off-diagonal terms is justified, in principle, simply by referring to the Independence Axiom and Information Axiom. That is particularly true when the off-diagonal terms cause a coupling in the design. The reference to the axioms, however, provides only nonspecific, abstract justification that is indiscriminately applied to all off-diagonal elements. On the other hand, it is evident that each of the off-diagonal terms presents different values when being removed. Given the differences, it is most likely that we must prioritize off-diagonal terms to be eliminated so that the effort can be most effectively spent. This section examines the value of off-diagonal element in two contexts. The first is the different impact of off-diagonal terms in decoupling a coupled design. Then, its effectiveness in reducing imaginary complexity is considered.

2.1 ELIMINATING AN OFF-DIAGONAL TERM TO DECOUPLE A COUPLED DESIGN

The Independence Axiom requires that a coupled design matrix be decoupled. This is achieved by modifying the current DP or adopting a new DP to ensure the troubling columns of the design matrix are replaced to have a desired structure. Thus, 'eliminating an off-diagonal term' is not quite accurate description of the decoupling activity. For the purpose of discussion, though, we assume that eliminating any off-diagonal term is always possible, and simply focus on the existence or disappearance of an off-diagonal term.

In decoupling a coupled design matrix, each off-diagonal element presents different value when eliminated. A coupled design matrix with certain structure may be decoupled by eliminating one critical off-diagonal term while a trivial solution may require eliminating more than one off-diagonal terms. Take, for example, the following coupled design matrix:

$$\begin{Bmatrix} FR1 \\ FR2 \\ FR3 \end{Bmatrix} = \begin{bmatrix} X & O & X \\ X & X & O \\ X & X & X \end{bmatrix} \begin{Bmatrix} DP1 \\ DP2 \\ DP3 \end{Bmatrix}$$

Figure 4. A 3x3, coupled design matrix

There are four off-diagonal elements, DM(2,1), DM(3,1), DM(3,2), and DM(1,3) to choose one or some combination from to eliminate. It is evident that eliminating DM(1,3) decouples the design matrix. Thus eliminating one off-diagonal term is sufficient to decouple the matrix. It should be noted, however, that among the four options, DM(1,3) is the only solution that can decouple the design by itself. Eliminating DM(3,1), DM(2,1), or DM(3,2) does not break the coupled relationship in the design. If DM(1,3) cannot be removed, decoupling requires eliminating the off-diagonal terms in either of the following pair: (DM(2,1), DM(3,1)) or (DM(3,1), DM(3,2)). Thus, it can be argued that the value of eliminating DM(1,3) is greater than that of DM(2,1), DM(3,1), or DM(3,2).

Understanding the different values of off-diagonal term and thereby prioritizing the task of eliminating them involves the following three questions.

a. Is there a coupling in the design matrix? If so, which off-diagonal terms constitute the coupling?
b. How many off-diagonal terms at minimum must be eliminated to decouple?
c. What are those off-diagonal terms?

In this section, we present an answer to the first question, and the second and third questions are discussed in section 3.

Existence of coupling in a design matrix

When the dimension of a design matrix is small, less than 4, or a design matrix is very sparse, one can easily determine the existence of coupling simply by visual inspection. The task becomes, however, quite challenging if a design matrix has a dimension larger than 5 and is reasonably populated. For example, it is not readily recognizable which of the following 5X5 matrices is coupled.

$$\begin{bmatrix} X & O & O & X & O \\ X & X & O & O & X \\ O & O & X & O & O \\ O & X & O & X & X \\ X & O & X & O & X \end{bmatrix} \qquad \begin{bmatrix} X & O & O & O & O \\ X & X & O & O & X \\ O & X & X & X & O \\ O & X & O & X & X \\ X & O & O & O & X \end{bmatrix}$$

design matrix (a) design matrix (b)

Figure 5. 5X5 design matrix: one on the left is a coupled matrix, and on the right is a decoupled.

Design matrix represents a network of interactions between a set of FRs and DPs. Off-diagonal terms, DM(i,j) represent a 'flow' of energy/material /information from DPi to FRj. If we let v_i to denote a design task DPi-FRi, then the two design matrices in Figure 5 have the following digraph representation.

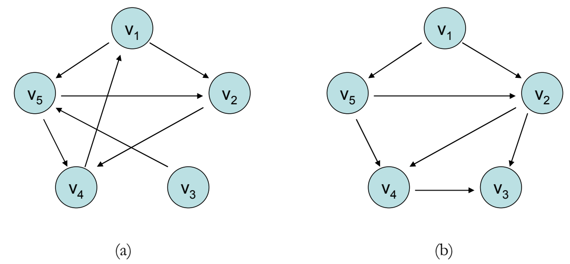


Figure 6. Digraph representation of the two design matrices in Figure 5

v_i in these graphs correspond to the diagonal elements of the deign matrix, and each of the directed arcs denotes an off-diagonal term. Now that a design matrix is represented in a digraph, many useful results from the graph theory can be readily applicable to the analysis of a design matrix.

With the digraph representation, the problem of determining the existence of a coupling becomes the problem of testing the existence of a cycle within the digraph. A cycle in a digraph is a path in which no vertex (and thus necessarily no edge) is repeated except for the start and end vertex. For example, $v_1 - v_2 - v_4 - v_1$ is a cycle. This cycle in the digraph indicates a coupling between FR1, FR2 and FR4. There is no cycle in the digraph (b), and the design matrix Figure 5(b) is thus a decoupled matrix.

Given a digraph, one can construct an algorithm to find, if any, a cycle in the graph. If the dimension of the problem is not too large, the following theorem [3][4] can be directly used instead of an algorithmic approach.

Theorem 1: Let A be a adjacency matrix of a size m×m. Then, $A^n(i,j)$ is the number of walks of length n from v_i to v_j.

Adjacency matrix of a digraph with m vertices is a m×m matrix where A(i,j) is the number of arcs from v_i to v_j. Adjacency matrix, A, and a design matrix, DM, has the following relationship:

$$A = DM^T - I \qquad (1)$$

where DM^T is a transpose of DM and I is m×m identify matrix. Theorem 1 says $A^n(i,i)$ is the number of walks of length n for v_i. For m-vertex digraph, the longest cycle would be at the length of m, i.e. visit all vertices once and come back to the origin vertex. Thus, if a design matrix of size m×m has a coupling, there exist non-zero diagonal terms in any of A, A^2, A^3, …, A^m. Using the design matrix (a) and (b) from Figure 6, Table 1 shows A^1 through A^5 for the two design matrices.

	A	A^2	A^3	A^4	A^5
	0 1 0 0 1	0 1 0 2 0	2 0 0 1 0	1 2 0 0 2	0 3 0 4 1
	0 0 0 1 0	1 0 0 0 0	0 1 0 0 1	0 1 0 2 0	2 0 0 1 0
(a)	0 0 0 0 1	0 1 0 1 0	1 0 0 1 0	1 1 0 0 1	0 2 0 2 1
	1 0 0 0 0	0 1 0 0 1	0 1 0 2 0	2 0 0 1 0	1 2 0 0 2
	0 1 0 1 0	1 0 0 1 0	1 1 0 0 1	0 2 0 2 1	2 1 0 3 0
	0 1 0 0 1	0 1 1 2 0	0 0 3 1 0	0 0 1 0 0	0 0 0 0 0
	0 0 1 1 0	0 0 1 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0
(b)	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0
	0 0 1 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0
	0 1 0 1 0	0 0 2 1 0	0 0 1 0 0	0 0 0 0 0	0 0 0 0 0

Table 1. Adjacency matrix up to 5th order to test the existence of a cycle

Table 1 shows that A3 and A4 of the design matrix (a) have non-zero diagonal elements whereas all the diagonal elements in A^1 through A^5 of design matrix (b) are zero, confirming that design matrix (a) is coupled and (b) is not.

Another piece of useful information we can infer from Table 1 is the total number of cycles in the digraph. It is easy to show that the sum of diagonal elements of A^k divided by k is the total number of closed walks of length k in the digraph. In the example (a), all the closed walks are cycles. Thus, for (a), there are two cycles of length 3 – (2+1+2+1)/2 – and one cycle of length 4, resulting in the total of three cycles. This is shown in Figure 7.

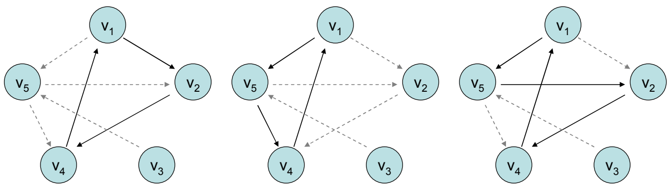


Figure 7. Two cycles of length 3 and one cycle of length 4

By visual inspection, it is easy to see that removing arc v_4v_1 eliminates all three cycles while the other arcs eliminates only one or two cycles that they participate. Recall that each arc in a digraph corresponds to an off-diagonal element in the design matrix, then eliminating the off-diagonal term (1,4) of the design matrix (a) completely decouples the design matrix.

Knowing the existence and the total number of coupling in a design matrix is the first step of building a complete optimal decoupling strategy. The question of finding the minimum set of off-diagonal elements will be discussed in section 3.

2.2 ELIMINATING AN OFF-DIAGONAL TERM TO SIMPLIFY THE INTERACTION STRUCTURE

The other aspect of the value an off-diagonal term is the reduction of imaginary complexity in a system. Imaginary complexity in Axiomatic Design theory is defined as the uncertainty in achieving FRs due to the lack of knowledge about the interactions between FRs and DPs. Interactions between FRs and DPs are represented by a design matrix. In the presence of such interaction, design process must be conducted such that the interaction is properly taken into account. Otherwise, it is likely that all or part of the design tasks need to be iterated, increasing uncertainty in satisfying FRs. We begin this section with a simple example to illustrate the concept of imaginary complexity. Take a full lower triangular matrix from Figure 1 (third matrix in the second row). While removing the off-diagonal term DM(2,1) uncouples FR1-FR2 and results in a 'simpler' interaction structure, removing DM(3,1) from the matrix does not effectively change the interaction structure as the design matrix remains fully decoupled.

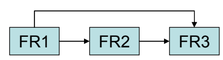
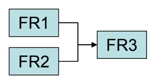
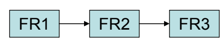
	Design Matrix	Flow chart	z	$C_I = -log_2 (z/n!)$
(a)	$\begin{bmatrix} X & O & O \\ X & X & O \\ X & X & X \end{bmatrix}$		1	2.58
(b)	$\begin{bmatrix} X & O & O \\ O & X & O \\ X & X & X \end{bmatrix}$		2	1.58
(c)	$\begin{bmatrix} X & O & O \\ X & X & O \\ O & X & X \end{bmatrix}$		1	2.58

Table 2. Off-diagonal elements have different impact on design improvement when eliminated from the design matrix: (a) flow diagram for a fully-lower triangular 3x3 design matrix (b) when DM(2,1) is removed, FR1 and FR2 become uncoupled (c) when DM(3,1) is removed, the fully-decoupled relationship between FR1, FR2 and FR3 remains unchanged. z, the number of acceptable sequences and C_I, imaginary complexity are also shown.

A flow chart, shown in Table 2, graphically illustrates the different impact DM(2,1) and DM(3,1) makes when they are removed. A square box in the figure represents a design task where FRi is achieved by DPi. An arrow between the boxes indicates the directed interaction/influence from one design task to another. The arrow from [FR1] to [FR2] is a visual representation of the design matrix element, DM(2,1). It can be argued, from the design iteration perspective, that eliminating DM(2,1) is a better option than DM(3,1) assuming both require same resource. The impact of an off-diagonal term can be quantified by using the definition of the imaginary complexity, C_I [1][5].

$$C_I = -\log_2 (z/m!) \qquad (2)$$

z is the number of acceptable sequences in carrying out m design tasks, and m! is the total number of sequences for m×m design problem. Imaginary complexity decreases as z increases, and z increases as off-diagonal elements are eliminated. Obviously, the value of z is determined by the structure of design matrix as we saw from Table 2. In order to construct any strategy regarding this aspect of design matrix, we must first clearly understand the relationship between z and off-diagonal terms. z is m! for an uncoupled design matrix as there is no particular sequence to be followed. On the other hand, z is zero for a coupled design. Only for a decoupled design, z is non-trivial and takes a value between 0 and m!. Thus, we focus on a decoupled design case in the following discussion.

Determining z of a decoupled m×m design matrix

z is the number of acceptable sequence in serially executing design tasks. An acceptable sequence is a sequence that does not incur an unnecessary iteration of design tasks. In order to avoid such iteration, design tasks should be sequenced to conform to all the precedence relationships. These precedence relationships are indicated by off-diagonal terms in a design matrix. An off-diagonal term DM(i,j) represents the precedence relationship: [FRj]> [FRi], i.e. FRj must be satisfied first to FRi. Finding z is to find a set of sequences that preserve all the precedence relationships.

It is more convenient to use the adjacency matrix notation, and for the following discussion, we use the adjacency matrix in place of a design matrix. For a adjacency matrix of m×m with one off-diagonal term A(i,j), half of the total m! sequences satisfy the precedence condition and the other half do not. Thus, m!/2 is the number of acceptable sequences. Let this set of sequences be $S_{i,j}$.

$$S_{i,j} = \{\text{design sequence of length m : design task i comes first to task j}\}$$

$S_{i,j}$ is defined for any off-diagonal element, and its size is always m!/2. It is straightforward to construct $S_{i,j}$ by permutation [6]. A set of acceptable sequences for a design matrix with multiple off-diagonal terms, A(p,q), A(u,v), … is obtained by

$$S^o = S_{p,q} \cap S_{u,v} \cap \ldots \qquad (3)$$

Then, the size of S^o is z. If $S^o = \varnothing$, then the matrix is coupled.

Although constructing $S_{i,j}$ and determining S^o is a straightforward process, it is not computationally efficient as m becomes larger: $S_{i,j}$ has m!/2 members and there are maximum of m(m-1)/2 $S_{i,j}$ to conduct an intersection operation. Equation (4) can help reduce the number of computation.

$$S_{i,j;j,k} = S_{i,j} \cap S_{j,k} \subset S_{i,k} \qquad (4)$$

It follows from the fact that the precedence relationship $\{i>j\} \wedge \{j>k\}$ implies $\{i>k\}$. In other words, given $\{i>j\}$ and $\{j>k\}$, $\{i>k\}$ is redundant. It is clear that equation (4) can be extended to longer chain of precedence relationships and thus can be generalized:

$$S_{i,i+1;i+1,i+2;\ldots;i+m,i+m+1} \subset S_{i+j,i+k} \qquad (5)$$
$$m \geq 1$$
$$0 \leq j \leq m$$
$$j+1 \leq k \leq m+1$$

Equation (5) suggests that in computing equation (3), one can reduce the computational steps by first finding the most expanded chains of precedence relationships and carry out equation (3) for only those that are included in the chains.

Alternatively, we can construct a set of acceptable sequences by enumerating all the sequences while continuously subjecting them to the precedence constraints. This process is illustrated with the following 5×5 adjacency matrix.

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (6)$$

Column 2 and 4 are zero-columns, indicating that design task 2 and 4 will be the first (initiating) task of all the sequences. On the other hand, row 3 and 5 are all zeros, and they will be the last (terminating) task of any sequence. There are six precedence constraints in this matrix: A(4,1), A(1,3), A(2,3), A(4,3), A(1,5), A(2,5). Among the six precedence constraints, we see that A(4,3) is redundant by equation (4): A(4,3) is implied in A(4,1)-A(1,3). These constraints are translated into three rules:

- Rule 1 – [FR1] must follow [FR4]
- Rule 2 – [FR3] must follow [FR1], [FR2] and [FR4]
- Rule 3 – [FR5] must follow [FR1] and [FR2]

Suppose we begin a search by choosing [FR2] as the initiating task. Then, we examine which of the remaining four tasks can be placed after [FR2]. [FR1] violates rule 1, [FR3] violates rule 2, and [FR5] violates rule 3. [FR4] does not have a rule associated with it, and thus is a legitimate task to be placed after [FR2]. After [FR4], [FR1] is the only option as placing [FR3] or [FR5] violates rule 2 and 3 respectively. Repeating this process, a complete tree for acceptable sequence is drawn to show z = 6 for this design matrix.

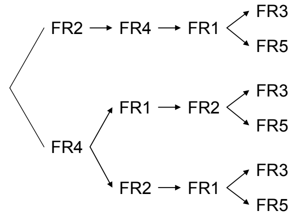


Figure 8. Acceptable sequence tree

3 OPTIMAL STRATEGY IN ELIMINATING OFF-DIAGONAL ELEMENTS

Section 2 presented groundwork on quantitative understanding of an off-diagonal element in two contexts: coupling and design iteration. Indeed, the works presented in section 2 can be directly used to search for an optimal set of off-diagonal terms to eliminate, in a brute force fashion. However, the brute force approach quickly becomes impractical as the size of a matrix becomes large and the matrix is reasonably populated (not sparse). Thus, it will be desirable to design a more efficient algorithm or heuristics to find an optimal strategy in choosing which off-diagonal elements to eliminate. This section describes the preliminary attempt to design an optimal decoupling strategy.

3.1 MINIMUM SET OF OFF-DIAGONAL TERMS THAT ELIMINATE COUPLING

When eliminating off-diagonal terms to decouple a design matrix, a trivial solution in selecting the target off-diagonal terms is to eliminate all of the off-diagonal terms above (or below) the diagonal to make the design matrix an apparent triangular matrix. In many cases, however, we can still decouple the matrix by eliminating smaller number of off-diagonal terms than the trivial solution. With an assumption that the number of off-diagonal terms to eliminate is proportional to the required engineering resource, a best solution is the one that requires minimum number of off-diagonal terms be eliminated. We call such best solution an optimal decoupling strategy. Before we discuss an optimal decoupling strategy, a few concepts from graph theory need to be introduced. First, we introduce the incidence matrix, B, and the cycle matrix, C.

The incidence matrix, B for a digraph with m vertices and n arcs – equivalent to m FR-DP pairs and n off-diagonal terms – is an m×n matrix where B(i,j) = 1 if arc j is directed away from a vertex v_i, B(i,j) = -1 if arc j is directed towards vertex v_i, and B(i,j) = 0 otherwise. The cycle matrix, C of a digraph is a matrix where C(i,j) = 1 if cycle Z_i of the digraph contains arc j directed in the same way as the orientation of Z_i, C(i,j) = -1 if Z_i contains arc j directed in the opposite way to the orientation of Z_i, and C(i,j) = 0 otherwise [3][4].

As an example, Table 3 shows the adjacency matrix, A, the incidence matrix, B, and the cycle matrix, C, for a digraph shown in Figure 9. B can be constructed directly from A once the arcs are labeled. Each digraph has a unique A and B to describe its structure. Obtaining C is not straightforward because we somehow need to identify all the cycles in the digraph. Fortunately, with the help of some of the findings in the graph theory, we can obtain C through a few intermediate steps from B [3]:

- Eliminate any one row from B to obtain B_r
- Identify a directed spanning tree in the digraph
- Partition B_r into B_c and B_t where B_t is a partition of B_r by including only those arcs in the spanning tree and the rest of B_r is B_c
- $C_t = -(B_t^{-1}B_c)^T$
- $C_f = [I_\mu : C_t]$ where I_μ is an identity matrix of dimension $\mu = e - m + 1$ where e is the number of arcs and m is the number of vertices
- Construct C by taking linear combinations of C_f

Thus, given B (or A, equivalently), we can construct C. In the example, B_r is obtained by removing the fifth row (v5 row). a6, a7, a8 and a9 can be chosen as the four arcs of a spanning tree, and then B_t is the 4×4 matrix between v1,v2,v3,v4 and a6,a7,a8,a9. B_c is the rest of B_r. μ is (9-5+1) = 5. Final outcome[1] of C is 7×9 matrix as shown in Table 3.

$$\begin{bmatrix} X & O & O & X & O \\ X & X & O & O & X \\ O & X & X & X & O \\ O & X & O & X & X \\ X & O & X & O & X \end{bmatrix}$$

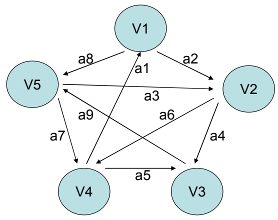


Figure 9. Design matrix and its digraph representation

A (adjacency matrix):

0	1	0	0	1
0	0	1	1	0
0	0	0	0	1
1	0	1	0	0
0	1	0	1	0

B (incidence matrix):

	a1	a2	a3	a4	a5	a6	a7	a8	a9
v1	-1	1	0	0	0	0	0	1	0
v2	0	-1	-1	1	0	1	0	0	0
v3	0	0	0	-1	-1	0	0	0	1
v4	1	0	0	0	1	-1	-1	0	0
v5	0	0	1	0	0	0	1	-1	-1

C (cycle matrix):

	a1	a2	a3	a4	a5	a6	a7	a8	a9
z1	1	0	0	0	0	0	1	1	0
z2	0	0	0	0	1	0	1	0	1
z3	1	1	0	0	0	1	0	0	0
z4	1	0	1	0	0	1	0	1	0
z5	0	0	1	1	0	0	0	0	1
z6	1	1	0	1	0	0	1	0	1
z7	0	0	1	0	1	1	0	0	1

Table 3. Adjacency matrix, Incidence matrix, and Cycle matrix

The cycle matrix C shown in Table 3 gives the complete picture of the coupling structure. There exist seven cycles in the digraph, and for each of the seven cycles, it tells you which arcs constitutes the cycle. In the design matrix terminology, it is equivalent to knowing the couplings present in the design matrix and also knowing exactly which off-diagonal terms constitute each of the couplings. Now that C is known, the task of finding optimal set of off-diagonal terms to decouple the design matrix is to find a minimum set of columns in C that, when summed, do not contain any zero entry. No zero entry in a (combination of) column indicates that the particular arc (or combination of arcs) is present in all the cycles, and thus removing it will destroy all the cycles. The minimum set of such columns can be easily done by performing exhaustive search for combinations of different columns from C starting from single column, two-columns, etc. to find the first set of columns with non-zero entries. Once the set is found, then the off-diagonal terms that correspond to the columns are the optimal set of off-diagonal terms to eliminate coupling. In the cycle matrix C shown in Table 3, the summation of the column a1 and column a9 makes a column without any zero entry. Therefore, minimum number of off-diagonal terms to remove to eliminate coupling from the design matrix is two, and they are DM(1,4) and DM(5,3).

To summarize, the overall process is as following:

[1] C, constructed from C_f, contains negative entries for some Zi, indicating that some of the arcs in cycle Zi are in the opposite direction. These cycles are called semi-cycles since they are not cycles. For convenience, we do not show those semi-cycles in C.

- Construct the adjacency matrix A to determine the existence of coupling
- If coupled, construct B from A
- Identify a directed spanning tree
- Given B(and thus B_r) and the spanning tree, construct C
- Search the combinations of the columns to find the first set of columns with non-zero entries when summed up
- DM(i,j) that correspond to the columns found are the minimum set of off-diagonal term that decouples the design matrix

By applying the above procedures, we can obtain the minimum set of off-diagonal terms to eliminate from a coupled design matrix to make it decoupled.

3.2 OPTIMAL SET OF OFF-DIAGONAL TERMS TO SIMPLIFY THE DESIGN INTERACTION STRUCTURE

We saw, in an example in Table 2, that eliminating the same number of off-diagonal terms can result in different amount of C_I reduction, depending on their structural context. In section 2.2, we presented the algorithms to compute z, the number of acceptable sequences for a decoupled design. In this section, we discuss a heuristic approach to determine an optimal set of off-diagonal terms to increase z. An optimal set here is defined for a given number of, say k, off-diagonal terms, and it is a set of k off-diagonal terms that increases z most.
Recall that three rules were identified from the adjacency matrix in equation (6). Three non-zero columns, column 1, 3, and 5, of the adjacency matrix become the three rules. Note that in Rule 2, A(4,3) is redundant by A(4,1) and A(1,3). Each of these rules is used when constructing an enumerating tree (Figure 10).

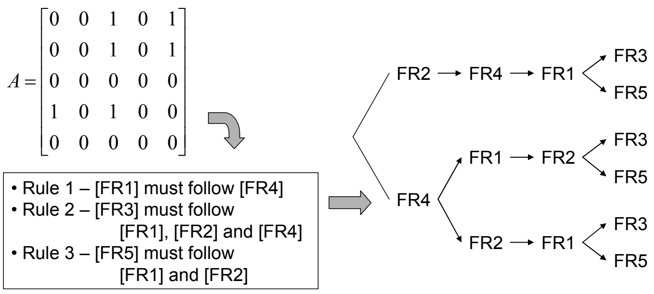


Figure 10. Example from section 2.2. Rules are derived from the adjacency matrix, and the enumeration tree is constructed. The tree shows z=6.

Absence of a precedence relationship for [FRi] adds certain degrees of freedom in choosing an acceptable [FRi] at a particular node in the tree, and results in a branch in the tree. The higher upstream exist the branches of a tree, the larger the number of leaf level branches will be. Giving additional degrees of freedom at the most upstream of the enumeration tree is equivalent to eliminating a rule for [FRi] thereby making [FRi]'s position completely flexible. For example, removing A(4,1) eliminates Rule 1 completely and thus gives a complete degree of freedom for [FR1]. As a result, it leads to additional branch at the first node of the tree. On the other hand, removing A(1,3) is not as effective as A(4,1). Although removing A(1,3) relaxes Rule 2, there still remains the precedence relationship for [FR3]. Consequently [FR3] cannot appear before [FR2] and [FR4], which means it can only appear at the third or lower branching node. Therefore, in removing a given number of off-diagonal terms, it is most effective to choose the ones that belong to the shortest rules. It is least effective to remove one from the longest rules. It is non-effective if a redundant off-diagonal term is removed. An equivalent statement in the matrix form is that the most effective set of off-diagonal terms to remove is a set of off-diagonal terms that renders the maximum number of columns to be a zero column. Figure 11 shows the effect of eliminating one off-diagonal term from the example matrix, A. Eliminating an off-diagonal term from the column with fewer non-zero entries is more effective in increasing z.

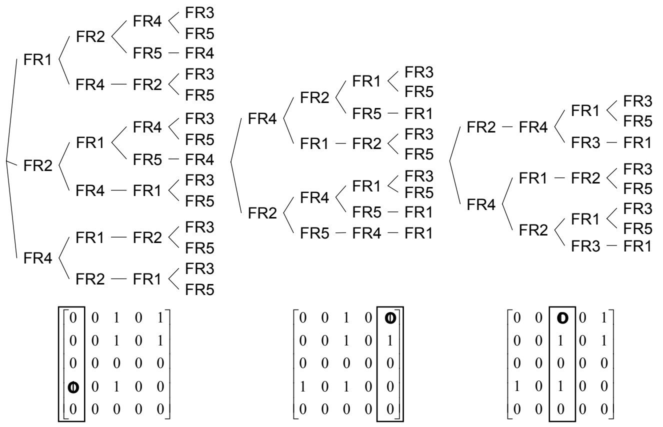


Figure 11. Removing one off-diagonal term from the matrix. Eliminating an off-diagonal term from the column with fewer non-zero entries is more effective in increasing z. z for (a) = 14, (b) = 9, and (c) = 8.

When applying this heuristics, care must be given to the redundant precedence relationship. For example, the precedence requirement between [FR3] and [FR4], A43, is redundant when both A41 and A13 are non-zero. However, once either of A41 or A13 becomes zero, A43 is not redundant any more. In other words, redundancy of a precedence requirement may change, and thus the redundancy of Aij must be continuously tested whenever a change is made to the matrix A.

4 CONCLUSIONS

Eliminating an off-diagonal term from a design matrix brings value in a few different contexts. For a coupled design, it can decouple the design matrix. For a decoupled design, it will increase the flexibility of design interaction structure and thereby reducing the imaginary complexity. While eliminating an off-diagonal term is always preferred in the Axiomatic Design framework, it must be justified in the context of the cost-benefit trade. Assuming the cost is proportional to the number of off-diagonal terms to eliminate, this paper presented methods to identify a minimum set of off-diagonal terms that achieve the two objectives – decoupling and imaginary complexity. For the decoupling problem, the cycle matrix, C, provides an effective means to identify the optimal set of off-diagonal terms. For the imaginary complexity problem, we presented a heuristic approach to select the optimal set. While the method we presented for the coupling problem guarantees the optimal solution, the heuristic method for the second problem needs further validation.

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